Multivariable Calculus

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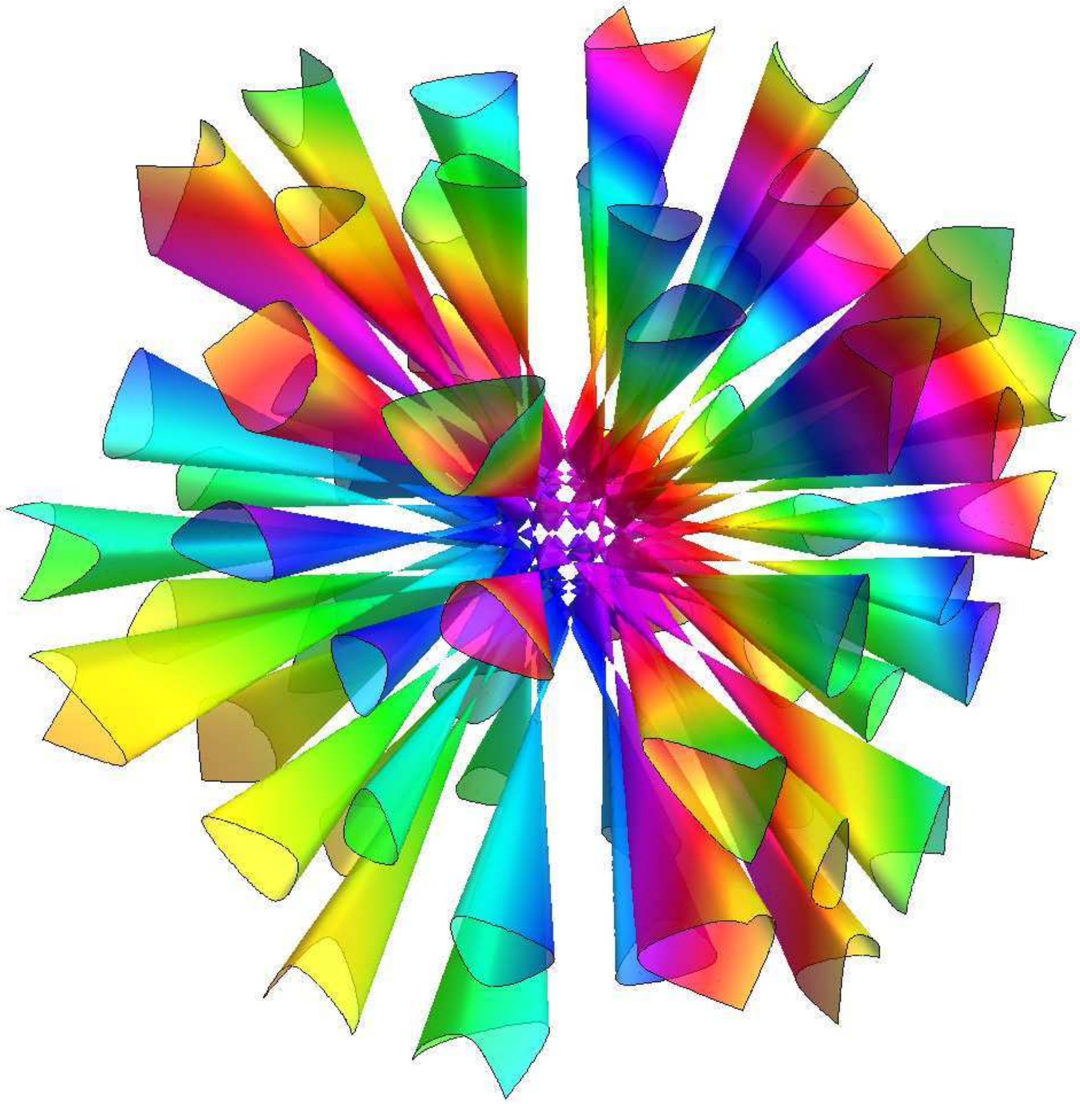
Math 21a, Fall 2012



These notes contain condensed "two pages per lecture" notes with essential information only. Remaining space was filled with problems.

Harvard Multivariable Calculus Math 21a, Fall 2012

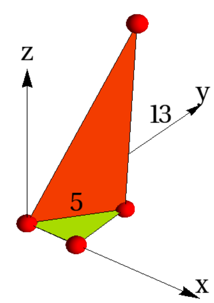
1: Geometry and Distance

A point in the **plane** has two **coordinates** $P = (x, y)$. A point in space is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates define 4 **quadrants** in the plane and 8 **octants** in space. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

1 Describe the location of the points $P = (1, 2, 3), Q = (0, 0, -5), R = (1, 2, -3)$ in words. **Possible Answer:** $P = (1, 2, 3)$ is in the positive octant of space, where all coordinates are positive. The point $R = (0, 0, -5)$ is on the negative z axis. The point $S = (1, 2, -3)$ is below the xy-plane. When projected onto the xy-plane it is in the first quadrant.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$. **Answer.** The midpoint is obtained by taking the average of each coordinate $M = (P + Q)/2 = (-1, 3, 6)$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.



This definition of Euclidean distance is motivated by the **Pythagorean theorem**. [1]

3 Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

A **circle** of radius r centered at $P = (a, b)$ is the collection of points in the plane which have distance r from P.

A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

4 Is the point $(3, 4, 5)$ outside or inside the sphere $(x-2)^2 + (y-6)^2 + (z-2)^2 = 16$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1 + 4 + 9}$ which is smaller than 4 the radius of the sphere. The point is inside.

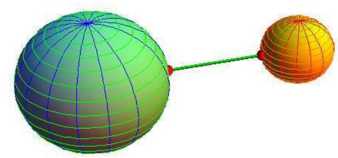
[1] It appears in an appendix to "Geometry" of "Discours de la méthode" from 1637, **René Descartes** (1596-1650). More about Descartes in "Descartes Secret Notebook" by Amir Aczel.

The **completion of the square** of an equation $x^2 + bx + c = 0$ is the idea to add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

[2]

5 Solve $2x^2 - 10x + 12 = 0$. **Answer.** The equation is equivalent to $x^2 + 5x = -6$. Adding $(5/2)^2$ on both sides gives $(x + 5/2)^2 = 1/4$ so that $x = 2$ or $x = 3$.

6 Find the center of the sphere $x^2 + 5x + y^2 - 2y + z^2 = -1$. **Answer:** Complete the square to get $(x+5/2)^2 - 25/4 + (y-1)^2 - 1 + z^2 = -1$ or $(x-5/2)^2 + (y-1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.



Al-Khwarizai

Rene Descartes

Distance between spheres

7 Find the set of points $P = (x, y, z)$ in space which satisfy $x^2 + y^2 = 9$. **Answer:** This is a cylinder of radius 3 around the z-axes parallel to the y axis.

8 What is $x^2 + y^2 = z^2$. **Answer**: this is the set of points for which the distance to the z axes is equal to the distance to the xy-plane. It must be a cone.

9 Find the distances of $P = (12, 5, 3)$ to the xy-plane. **Answer:** 3. Find the distance of $P = (12, 5, 0)$ to z axes. **Answer:** 13.

10 Describe $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$. **Answer:** Complete the square to get a sphere $(x + 2)^2 + (y - 8)^2 + (z + 5)^2 = 36$ with center $(-2, 8, -5)$ and radius 6.

11 Describe the set $xz = 0$. **Answer:** We either must have $x = 0$ or $z = 0$. The set is a union of two coordinate planes.

12 Find an equation for the set of points which have the same distance to $(1, 1, 1)$ and $(0, 0, 0)$. **Answer:** $(x-1)^2 + (y-1)^2 + (z-1)^2 = x^2 + y^2 + z^2$ gives $-2x + 1 - 2y + 1 - 2z + 1 = 0$ or $2x + 2y + 2z = 3$. This is the equation of a plane.

13 Find the distance between the spheres $x^2 + (y-12)^2 + z^2 = 1$ and $(x-3)^2 + y^2 + (z-4)^2 = 9$. **Answer:**The distance between the centers is $\sqrt{3^2 + 4^2 + 12^2} = 13$. The distance between the spheres is $13 - 3 - 1 = 9$.

[2]Due to **Al-Khwarizmi** (780-850) in *"Compendium on Calculation by Completion and Reduction"* The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Source book, Ed Victor Katz, contains translations of some of this work.

2: Vectors and Dot product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{v} = \langle x - a, y - b - z - c\rangle$. It points from P to Q and we write also $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in a vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ are called the **components** of $\vec{v}$.

Similar definitions hold in two dimensions, where vectors have two components. Vectors can be drawn **everywhere** in space but two vectors with the same components are considered **equal**. [1]

The **addition** of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2, u_3\rangle + \langle v_1, v_2, v_3\rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$. The **scalar multiple** $\lambda\vec{u} = \lambda\langle u_1, u_2, u_3\rangle = \langle \lambda u_1, \lambda u_2, \lambda u_3\rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1)\cdot\vec{v}$.

The addition and scalar multiplication of vectors satisfy the laws you know from **arithmetic**. **commutativity** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, **associativity** $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ and $r*(s*\vec{v}) = (r*s)*\vec{v}$ as well as **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$ and $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$, where $*$ is scalar multiplication.

The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**.

1 $|\langle 3, 4\rangle| = 5$ and $|\langle 3, 4, 12\rangle| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $\langle 3/5, 4/5\rangle$ and $\langle 3/13, 4/13, 12/13\rangle$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c\rangle$ and $\vec{w} = \langle p, q, r\rangle$ is defined as $\vec{v}\cdot\vec{w} = ap + bq + cr$.

The dot product determines distance and distance determines the dot product.

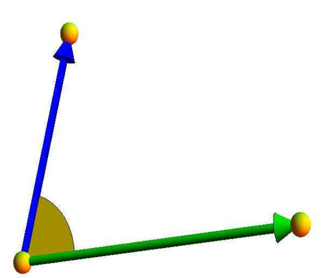
Proof: Lets write $v = \vec{v}$ in this proof. Using the dot product one can express the length of v as $|v| = \sqrt{v\cdot v}$. On the other hand, $(v + w)\cdot(v + w) = v\cdot v + w\cdot w + 2(v\cdot w)$ allows to solve for $v\cdot w$:

$$v\cdot w = (|v + w|^2 - |v|^2 - |w|^2)/2 .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v}\cdot\vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. We can assume $|w| = 1$ after scaling the equation. Now plug in $a = v\cdot w$ into the equation $0 \leq (v-aw)\cdot(v-aw)$ to get $0 \leq (v-(v\cdot w)w)\cdot(v-(v\cdot w)w) = |v|^2+(v\cdot w)^2-2(v\cdot w)^2 = |v|^2-(v\cdot w)^2$ which means $(v\cdot w)^2 \leq |v|^2$.

The Cauchy-Schwarz inequality allows us to define what an "angle" is.



The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cos(\alpha)$.

[1] We cover 2300 years of math from Pythagoras (500 BC), Al Kashi (1400), Cauchy (1800) to Hamilton (1850).

Al Kashi's theorem: A triangle ABC with side lengths a, b, c and angle α opposite to c satisfies $a^2 + b^2 = c^2 + 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

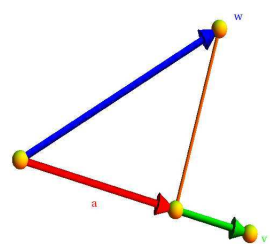
Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2+\vec{v}^2+2\vec{u}\cdot\vec{v} \leq \vec{u}^2+\vec{v}^2+2|\vec{u}\cdot\vec{v}| \leq \vec{u}^2+\vec{v}^2+2|\vec{u}|\cdot|\vec{v}| = (|\vec{u}|+|\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$.

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. Quod erat demonstrandum. [2]

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$.



2 Find the projection of $\vec{v} = \langle 0, -1, 1 \rangle$ onto $\vec{w} = \langle 1, -1, 0 \rangle$. **Ansser:** $\mathrm{P}(\vec{v}) = \langle 1/2, -1/2, 0 \rangle$.

3 A wind force $\vec{F} = \langle 2, 3, 1 \rangle$ is applied to a car which drives in the direction of the vector $\vec{w} = \langle 1, 1, 0 \rangle$. Find the projection of $\vec{F}$ onto $\vec{w}$, the force which accelerates or slows down the car. **Answer**: $\vec{w}(\vec{F} \cdot \vec{w}/|\vec{w}|^2) = \langle 5/2, 5/2, 0 \rangle$.

4 How can we visualize the dot product? **Answer**: Difficult task but lets try. The absolute value of the dot product is the length of the projection. The dot product is positive if $\vec{v}$ and $\vec{w}$ form an acute angle, negative if that angle is obtuse.

5 Given $\vec{v} = \langle 2, 1, 2 \rangle$ and $\vec{w} = \langle 3, 4, 0 \rangle$. Find a vector which is in the plane defined by $\vec{v}$ and $\vec{w}$ and which bisects the angle between these two vectors. **Answer**. Normalize the two vectors to make them unit vectors then add them to get $\langle 13, 17, 10 \rangle/15$.

6 Given two vectors $\vec{v}, \vec{w}$ which are perpendicular. Under which condition is $\vec{v}+\vec{w}$ perpendicular to $\vec{v} - \vec{w}$? **Answer**: Find the dot product of $\vec{v} + \vec{w}$ with $\vec{v} - \vec{w}$ and set it zero.

7 Is the angle between $\langle 1, 2, 3 \rangle$ and $\langle -15, 2, 4 \rangle$ acute or obtuse? **Answer**: the dot product is 1. Cute!

[2] Short QED. We have just proven Pythagoras and AlKashi. Distance and angle were defined, not deduced.

3: Cross product

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1 w_2 - v_2 w_1 = \det \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle .$$

[1]

To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

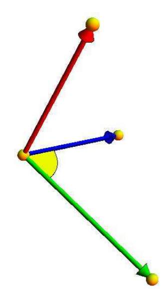
which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$.

1 The cross product of $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$ is a scalar and $5 - 8 = -3$.

2 The cross product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 1 \rangle$ is the vector $\langle -13, 11, -3 \rangle$.

The cross product $\vec{v} \times \vec{w}$ is anti-commutative. The resulting vector is orthogonal to both $\vec{v}$ and $\vec{w}$.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.



The **sin** formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof: We verify the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

[1] It was Hamilton who described in 1843 first a multiplication $*$ of 4 vectors. It contains intrinsically both dot and cross product because $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-vw, v \times w)$.

The definition shot fits with our intuition: $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. The area does not change if we rotate the vectors around in space because both length and angle stay the same. Area also is linear in each of the vectors v and w. If we make v twice as long, then the area gets twice as large.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda\vec{w}$ for some real λ.

Proof. This follows immediately from the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

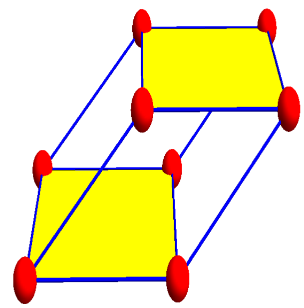
The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel even so sometimes one calls this anti-parallel.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

Proof. The area of the triangle is $ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta)$ Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

3 If $\vec{v} = \langle a, 0, 0\rangle$ and $\vec{w} = \langle b\cos(\alpha), b\sin(\alpha), 0\rangle$, then $\vec{v} \times \vec{w} = \langle 0, 0, ab\sin(\alpha)\rangle$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$ and the **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$.



These definitions fit intuition: the value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is indeed the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand.

4 **Problem:** Find the volume of a **cuboid** of width a length b and height c. **Answer**. The cuboid is a parallelepiped spanned by $\langle a, 0, 0\rangle$ $\langle 0, b, 0\rangle$ and $\langle 0, 0, c\rangle$. The triple scalar product is abc.

5 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that it is spanned by the vectors $\vec{u} = \langle 1, 2, 1\rangle$, $\vec{v} = \langle 3, 2, 1\rangle$, and $\vec{w} = \langle 0, 3, 1\rangle$. We get $\vec{v} \times \vec{w} = \langle -1, -3, 9\rangle$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

6 **Problem:** find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$. To do so find a vector $\vec{n} = \langle a, b, c\rangle$ normal to the and noting $(\vec{x} - \vec{OP}) \cdot \vec{n} = 0$. **Answer:** A normal vector $\vec{n} = \langle 1, -2, 2\rangle = \langle a, b, c\rangle$ of the plane $ax + by + cz = d$ is obtained as the cross product of $\vec{PQ}$ and $\vec{RQ}$ With $d = \vec{n} \cdot \vec{OP} = \langle 1, -2, 2\rangle \cdot \langle 1, 2, 3\rangle = 3$, we get the equation $x - 2y + 2z = 3$.

4: Lines and Planes

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c \rangle$ define the **line**

$$L = \{ \langle p, q, r \rangle + t\langle a, b, c \rangle, t \in \mathbf{R} \} .$$

The line is obtained by adding a multiple of the vector $\vec{v}$ to the vector $\vec{OP} = \langle p, q, r \rangle$. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". For $t = 0$, we are at P and for $t = 1$ we are at $\vec{OP} + \vec{v}$.

If t is restricted to the **parameter interval** $[s, u]$, then $L = \{ \langle p, q, r \rangle + t\langle a, b, c \rangle, s \leq t \leq u \}$ is a **line segment** connecting $\vec{r}(s)$ with $\vec{r}(u)$.

1 **Problem.** Get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$. **Solution.** with $\vec{v} = \vec{PQ} = \langle 1, 3, 4 \rangle$ we get get $L = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 1, 3, 4 \rangle; \}$ which is $\vec{r}(t) = \langle 1 + t, 1 + 3t, 2 + 4t \rangle$. Since $\langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 1, 3, 4 \rangle$ consists of three equations $x = 1+2t, y = 1+3t, z = 2+4t$ we can solve each for t to get $t = (x - 1)/2 = (y - 1)/3 = (z - 2)/4$.

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = \langle a, b, c \rangle$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

2 Find the symmetric equations for the line through the two points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$ **Solution.** first first form the parametric equations $\langle x, y, z \rangle = \langle 0, 1, 1 \rangle + t\langle 2, 2, 3 \rangle$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$ and solve for t to get $x/2 = (y - 1)/2 = (z - 1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{ \vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.

4 An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 2, 4, 6 \rangle + s\langle 1, 0, -1 \rangle \}$. This is called the **parametric description** of a plane.

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = \langle a, b, c \rangle$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have just shown

The equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$

$$ax + by + cz = d \ ,$$

where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ and d is obtained by plugging in $\vec{x}_0$.

5 **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = \langle 1, 2, 0 \rangle$ and $\vec{w} = \langle 2, 2, 2 \rangle$. We have $\vec{n} = \langle 4, -2, -2 \rangle$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

6 **Problem:** Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. **Answer:** find the angle between $\vec{n} = \langle 1, 1, 0 \rangle$ and $\vec{m} = \langle 1, 1, 1 \rangle$. It is $\arccos(2/\sqrt{6})$.

Finally, lets look at some distance functions.

The distance between P and $\Sigma : \vec{n} \cdot \vec{x} = d$ containing Q is $d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$.

Proof. Project PQ onto $\vec{n}$.

The distance between P and the line L is $d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$.

Proof: the area of the parallelogram spanned by PQ and $\vec{u}$ divided by the base length $|\vec{u}|$.

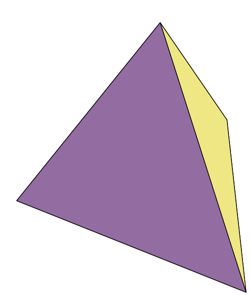
The lines $L : \vec{r}(t) = Q + t\vec{u}, M : \vec{s}(t) = P + t\vec{v}$ have distance $d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$.

Proof. Project PQ onto $\vec{n} = \vec{u} \times \vec{v}$.

The distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ is $d(\Sigma, \Pi) = \frac{|e-d|}{|\vec{n}|}$.

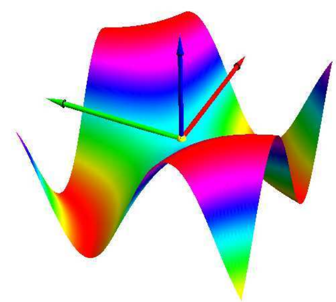
Proof: If P is on the first and Q on the second plane, the distance is the scalar projection of $\vec{PQ}$ onto $\vec{n}$. Note that $\vec{PQ} \cdot \vec{n} = d - e$.

7 A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3)$,$P_2 = (0, \sqrt{8}, -1)$, $P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.



Lecture 5: Functions

A **function of two variables** $f(x,y)$ is a rule which assigns to two numbers x,y a third number $f(x,y)$. For example, the function $f(x,y) = x^2y+2x$ assigns to $(3,2)$ the number $3^2 2 + 6 = 24$. The **domain** D of a function is set of points where f is defined, the range is $\{f(x,y) \mid (x,y) \in D\}$. The **graph** of $f(x,y)$ is the surface $\{(x,y,f(x,y)) \mid (x,y) \in D\}$ in space Graphs allow to visualize functions.

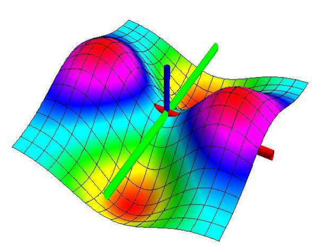


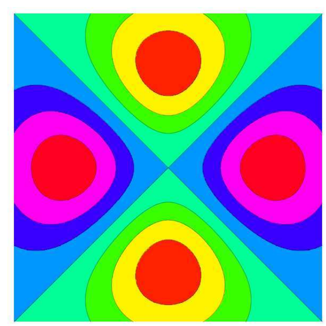
1 The graph of $f(x,y) = \sqrt{1-(x^2+y^2)}$ on the domain $D = \{x^2+y^2 < 1\}$ is a half sphere. The range is the interval $[0,1]$.

The set $f(x,y) = c = \text{const}$ is called a **contour curve** or **level curve** of f. For example, for $f(x,y) = 4x^2+3y^2$, the level curves $f = c$ are ellipses if $c > 0$. The collection of all contour curves $\{f(x,y) = c\}$ is called the **contour map** of f.

2 For $f(x,y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The curve $x^2 - y^2 = 1$ is made of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The curve $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

3 For $f(x,y) = (x^2-y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2} = c$. but we can draw the curves with the computer:



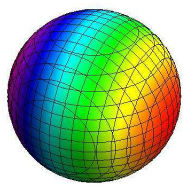


A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. We can visualize it by **contour surfaces** $g(x,y,z) = c$, where c is constant. It is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

4 For $g(x,y,z) = z - f(x,y)$, the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2 + y^2$ of the function $f(x,y) = x^2 + y^2$ which is a paraboloid. Most surfaces $g(x,y,z) = c$ are not graphs.

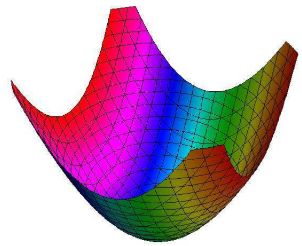
5 If $f(x, y, z)$ is a polynomial and $f(x, x, x)$ is quadratic in x, then $\{f = c\}$ is a **quadric**.

Sphere



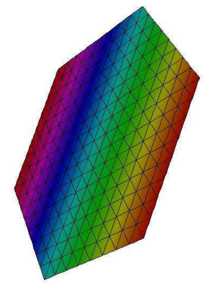
$x^2 + y^2 + z^2 = 1$

Paraboloid



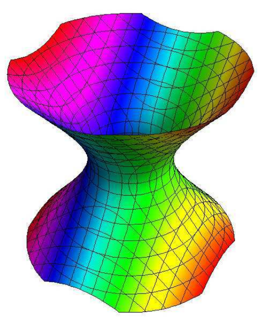
$x^2 + y^2 - c = z$

Plane



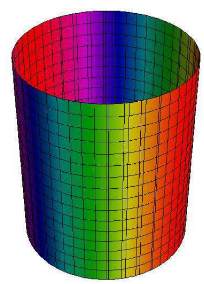
$ax + by + cz = d$

One sheeted Hyperboloid



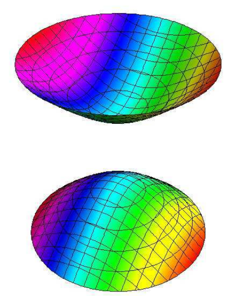
$x^2 + y^2 - z^2 = 1$

Cylinder



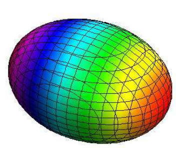
$x^2 + y^2 = r^2$

Two sheeted Hyperboloid



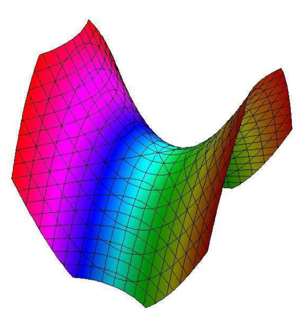
$x^2 + y^2 - z^2 = -1$

Ellipsoid



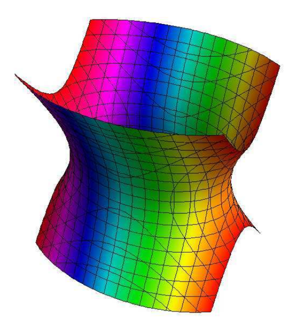
$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Hyperbolic paraboloid



$x^2 - y^2 + z = 1$

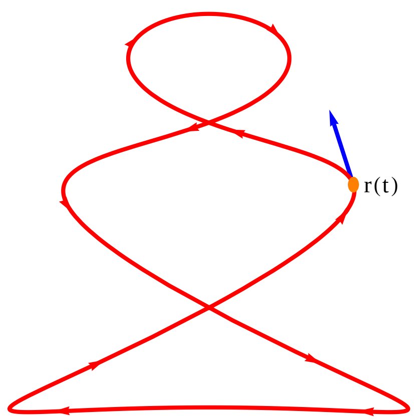
Elliptic hyperboloid

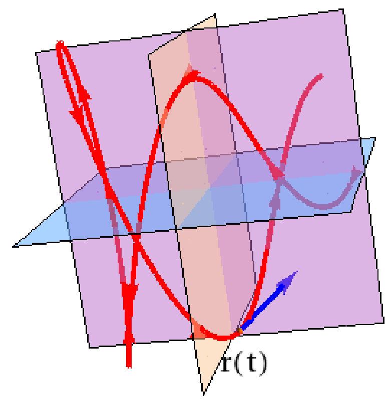


$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$

Lecture 6: Curves

A **parametrization** of a planar curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. The parametrization of a space curve is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. The **image** of r is a **parametrized curve** in space.





We always think of the **parameter** t as **time**. For a fixed t, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As t varies, the end point of this vector moves along a curve. The parametrization contains more information about the curve then the curve. It tells also how fast and in which direction we trace the curve.

1 The parametrization $\vec{r}(t) = \langle 1 + 3\cos(t), 3\sin(t) \rangle$ is a **circle** of radius 3 centered at $(1, 0)$

2 $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ defines a **Lissajous curve** example.

3 If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t) \rangle$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola.

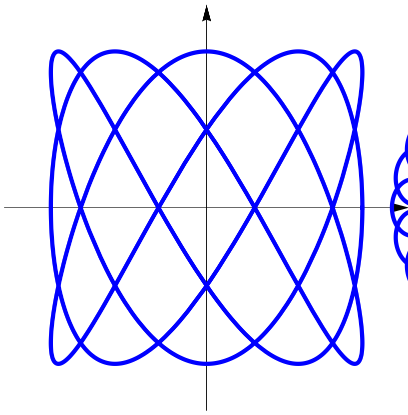
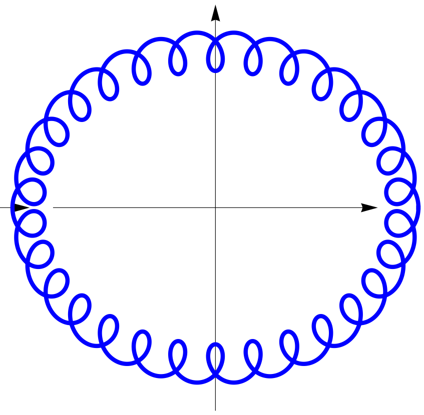
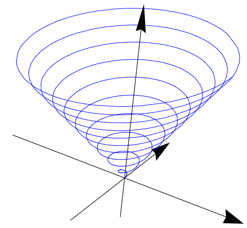
4 With $x(t) = 2\cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows an **ellipse** $x(t)^2/4 + y(t)^2 = 1$.

5 The space curve $\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$ traces a **helix** with increasing radius.

6 If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$ is the same curve as before but the **parameterization** has changed.

7 With $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$ it is traced in the **opposite direction**.

8 With $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle + 0.1\langle \cos(17t, \sin(17t) \rangle$ we have an example of an **epicycle**, where a circle turns on a circle. It was used in the **Ptolemaic geocentric system** which predated the Copernican system still using circular orbits and then the modern Keplerian system, where planets move on ellipses and which can be derived from Newton's laws.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

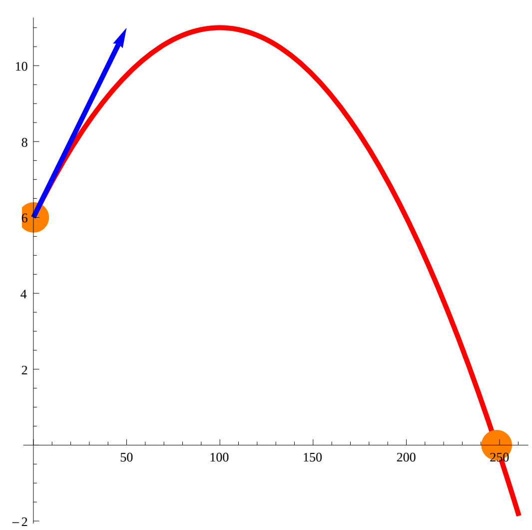
Any vector parallel to $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

The **addition rule** in one dimension $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.
The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ at all times as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

The **free fall** is the case when acceleration is constant. In particular, if $\vec{r}''(t) = \langle 0, 0, -10 \rangle$, $\vec{r}'(0) = \langle 0, 1000, 2 \rangle, \vec{r}(0) = \langle 0, 0, h \rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2 \rangle$.

If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2$.



7: Arc length and curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\;dt$ is called the **arc length of the curve**. In space the length is $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\;dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = \langle R\cos(t), R\sin(t)\rangle$ parameterized by $0 \le t \le 2\pi$ is 2π because the speed $|\vec{r}\,'(t)|$ is constant and equal to R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ has velocity $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1\rangle$ and constant speed $|\vec{r}\,'(t)| = \langle -\sin(t), \cos(t), 1\rangle = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2\rangle$$

for $1 \le t \le 2$? **Answer:** Because $\vec{r}\,'(t) = \langle 1, 1/t, t\rangle$, we have $\vec{r}\,'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\;dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. Because the curve constructed in such a way that the arc length can be computed, we call it "opportunity".

4 Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 6t, t^3\rangle$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle, 0 \le t \le 2\pi$? **Answer:** We have $|\vec{r}\,'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi} (3/2)\sin(2t)\;dt = 6$.

6 Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ for $-1 \le t \le 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. The speed is $|\vec{r}(t)| = \sqrt{t^2 + t^4}$. Because $\int x\sqrt{1+x^2}\;dx = (1+x^2)^{3/2}/3$, the arc length integral can be evaluated as $\int_{-1}^1 |t|\sqrt{1+t^2}\;dx = 2\int_0^1 t\sqrt{1+t^2}\;dt = 2(1+t^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = \langle t + \sin(t), \cos(t)\rangle$ parameterized by $0 \le t \le 2\pi$. We have $|\vec{r}\,'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\;dt$. A **substitution** $t = 2u$ gives $L = \int_0^{\pi} \sqrt{2 + 2\cos(2u)}\;2du = \int_0^{\pi} \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\;2du = \int_0^{\pi} \sqrt{4\cos^2(u)}\;2du = 4\int_0^{\pi} |\cos(u)|\;du = 8$.

8 Find the arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t)\rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1\;,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. The answer is $\int_{-1}^1 \cosh(t)\;dt = 2\sinh(1)$.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t\langle -\sin(t), \cos(t) \rangle$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\,dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 We do not always have a closed formula for the arc length of a curve. The length of the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ leads to $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\,dt$ which needs to be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the magnitude of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve turns sharply. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. If $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = \langle t, f(t) \rangle$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by $\vec{N}$ and $\vec{B}$ is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because B is automatically normal to T and N, we have shown:

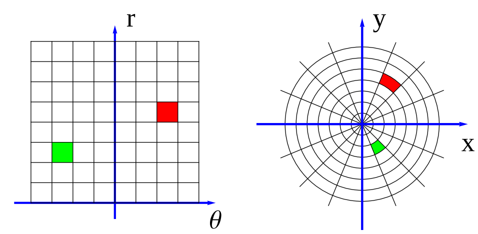
The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

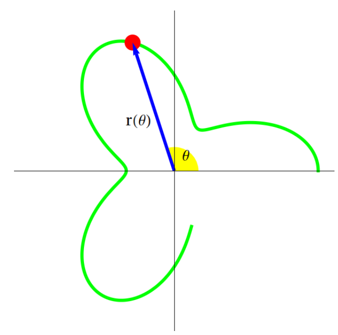
8: Polar and spherical coordinates

A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have the relation $(x, y) = (r\cos(\theta), r\sin(\theta))$.



The formula $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ have the same θ value. To get the correct θ, one can choose $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ when (x, y) is on the positive y axes, and add π on the left plane including the negative y axes.

A curve given in polar coordinates as $r(\theta) = f(\theta)$ is called a **polar curve**. It can in Cartesian coordinates be described as $\vec{r}(t) = \langle f(t)\cos(t), f(t)\sin(t)\rangle$.



1 Describe the curve $r = \theta$ in Cartesian coordinates. **Solution** A formal substitution gives $\sqrt{x^2 + y^2} = \arctan(y/x)$ but we can do better. Remember that for the curve $r(t) = \langle r\cos(t), r\sin(t)$ we have exactly the relation $r = t$. The curve is a spiral.

2 What is the curve $r = |2\sin(\theta)|$? **Solution** Lets ignore the absolute value for a moment and look at $r^2 = 2r\sin(\theta)$. This can be written as $x^2 + y^2 = 2y$ which is $x^2 + y^2 - 2y + 1 = 1$. The curve is a circle of radius 1 centered at $(0, 1)$. Since we have the absolute value, the radius at θ and $\theta + \pi$ is the same and add the circle of radius 1 centered at $(0, -1)$.

If we represent points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

we speak of **cylindrical coordinates**.

Here are some surfaces described in cylindrical coordinates:

3 $r = 1$ is a **cylinder**,

4 $r = |z|$ is a **double cone**

5 $\theta = 0$ is a **half plane**

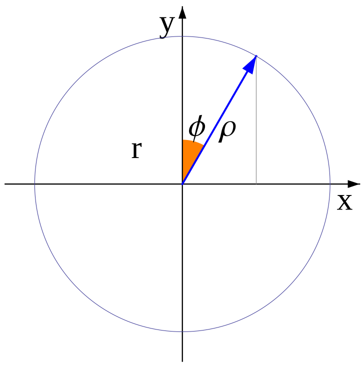
6 $r = \theta$ is a **rolled sheet of paper**

7 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

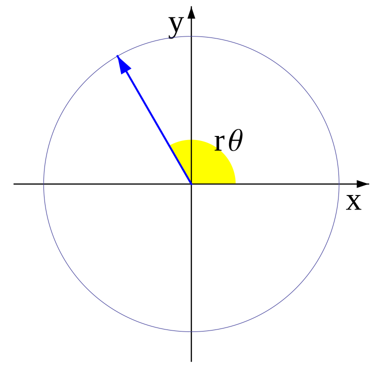
Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ. The first angle θ is the polar angle in polar coordinates of the xy coordinates and ϕ is the angle between the vector $\vec{OP}$ and the z-axis. The relation is

$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

There are two important figures to see the connection. The distance to the z axes $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture the rz-plane, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture the xy-plane.



$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi), \\ y &= \rho\sin(\theta)\sin(\phi), \\ z &= \rho\cos(\phi) \end{aligned}$$



Here are some level surfaces described in spherical coordinates:

8 $\rho = 1$ is a **sphere**,

9 The surface $\phi = \pi/2$ is a **single cone**

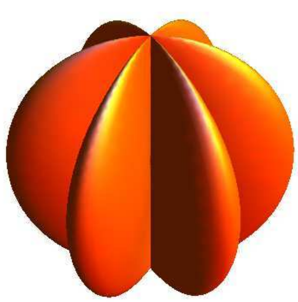
10 The surface $\sin(\theta) = \cos(\phi)$ is a **plane**.

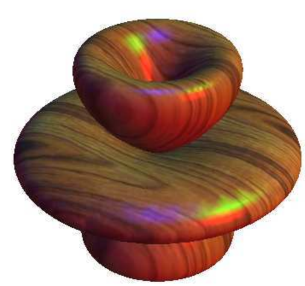
11 $\rho = \phi$ is an **apple shaped surface**

12 $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

13 Write $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates! Answer: $r^2 - 5r\cos(\theta) = z^2$.

14 Match the surfaces with $\rho = |\sin(3\phi)|, \rho = |\sin(3\theta)|$ in spherical coordinates (ρ, θ, ϕ). It helps to see this in the rz plane or the xy plane.





9: Parametrized surfaces

Beside an implicit equation $g(x, y, z) = 0$, the **parametrization** is a second, fundamentally different way to describe a surface.

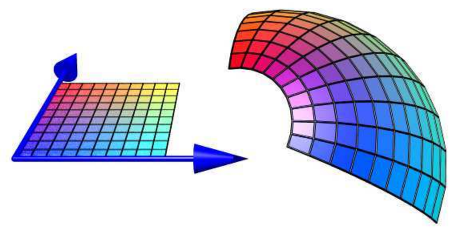
A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle ,$$

where $x(u, v), y(u, v), z(u, v)$ are three functions of two variables.

Because two parameters u and v are involved, the map $\vec{r}$ from the plane to space is also called uv-**map**.

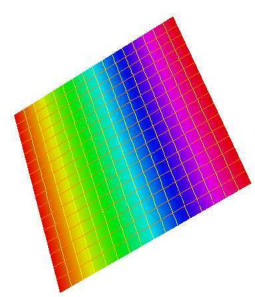
A **parametrized surface** is the image of the uv-map. The domain of the uv-map is called the **parameter domain**.



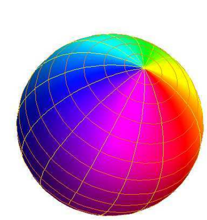
If the first parameter u is kept constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, for constant v, the map $u \mapsto \vec{r}(u, v)$ traces a curve on the surface. These curves are called **grid curves**.

A computer draws surfaces using grid curves. The world of parametric surfaces is intriguing. It can be explored with the help of a computer. Keep in mind the following 4 important examples. They cover a wide range of cases.

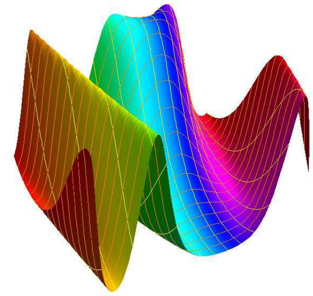
I **Planes**. Parametric: $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$
Implicit: $ax + by + cz = d$. Parametric to Implicit: find the normal vector $\vec{n} = \vec{v} \times \vec{w}$.
Implicit to Parametric: find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.



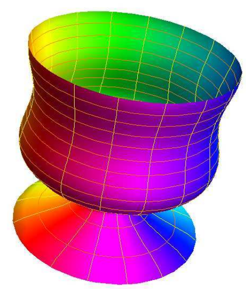
II **Spheres**: Parametric: $\vec{r}(u,v) = \langle a,b,c\rangle + \langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)\rangle$.
Implicit: $(x-a)^2+(y-b)^2+(z-c)^2=\rho^2$.
Parametric to implicit: read off the radius and the center
Implicit to parametric: find the center (a,b,c) and the radius r possibly by completing the square.



III **Graphs**:
Parametric: $\vec{r}(u,v) = \langle u,v,f(u,v)\rangle$
Implicit: $z - f(x,y) = 0$. Parametric to implicit: think about $z = f(x,y)$
Implicit to parametric: use x and y as the parameterizations.



IV **Surfaces of revolution**:
Parametric: $\vec{r}(u,v) = \langle g(v)\cos(u), g(v)\sin(u), v\rangle$
Implicit: $\sqrt{x^2+y^2} = r = g(z)$ can be written as $x^2+y^2=g(z)^2$.
Parametric to implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to parametric: again, the function g is the key link.



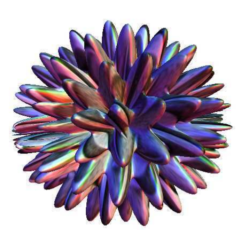
1 Describe the surface $\vec{r}(u,v) = \langle v^5\cos(u), v^5\sin(u), v\rangle$, where $u \in [0,2\pi]$ and $v \in \mathbf{R}$. **Solution.** It is a surface of revolution. We have $r = v^5 = z^5$. Draw this in the rz-plane. You see that r is small the origin making the surface pointy like a needle at the tip.

2 Find a parametrization for the plane which contains the three points $P=(3,7,1)$,$Q=(6,2,1)$ and $R=(0,3,4)$. **Solution.** Take $\vec{r}(s,t) = \vec{OP} + s\vec{QP} + t\vec{RP}$. $\vec{r}(s,t) = (3-3s-3t, 7-5s-4t, 1+3t)$.

3 Parametrize the lower half of the ellipsoid $x^2/4+y^2/9+z^2/25=1, z<0$. **Solution**. One solution is to solve for z and write it as a graph $\vec{r}(u,v) = \langle u,v,-\sqrt{25-25u^2/4-25v^2/9}\rangle$. We can also deform a sphere $\vec{r}(\theta,\phi) = \langle 2\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 5\cos(\phi)\rangle$.

4 Parametrize the upper half of the hyperboloid $x^2+y^2/4-z^2=-1$. **Solution.** The round hyperboloid, where $r^2=z^2-1$ is given by $\vec{r}(\theta,z) = \langle\sqrt{z^2-1}\cos(\theta), \sqrt{z^2-1}\sin(\theta), z\rangle$. Deform this now to get $\vec{r}(\theta,z) = \langle\sqrt{z^2-1}\cos(\theta), 2\sqrt{z^2-1}\sin(\theta), z\rangle$.

5 Describe the surface $\vec{r}(u,v) = \langle 2+\sin(13u)+\sin(17v)\rangle\langle\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$. **Solution.** It looks a bit like a hedgehog. It is an example of a "bumpy sphere". The radius ρ is a function of the angles. In spherical coordinates, we have $\rho = (2+\sin(13\theta)+\sin(17\phi))$.



10: Continuity

A function $f(x, y)$ with domain R is is **continuous at a point** $(a, b) \in R$ if $f(x, y) \to f(a, b)$ whenever $(x, y) \to (a, b)$. The function f is **continuous on** R, if f is continuous for every point (a, b) on R.

1 a) $f(x, y) = x^2 + y^4 + xy + \sin(y + \sin\sin\sin\sin(x)^2$ is continuous on the entire plane. It is built up from functions which are continuous using addition, multiplication or composition of functions which are all continuous everywhere.

2 $f(x, y) = 1/(x^2+y^2)$ is continuous everywhere except at the origin, where it is not defined.

3 $f(x, y) = y + \sin(x)/|x|$ is continuous away from $x = 0$. At every point $(0, y)$ it is discontinuous. $f(1/n, y) \to y + 1$ and $f(-1/n, y) \to y - 1$ for $n \to \infty$.

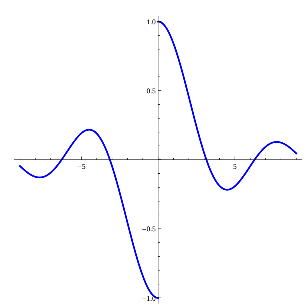
4 $f(x, y) = \sin(1/(x + y))$ is continuous except on the line $x + y = 0$.

5 $f(x, y) = (x^4 - y^4)/(x^2 + y^2)$ is continuous at $(0, 0)$. You can divide out $x^2 + y^2$ to see that the function is equivalent to $x^2 - y^2$ away from $(0, 0)$. After defining $f(0, 0) = 0$ we see that the function is continuous.

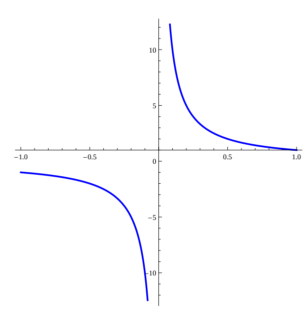
6 There are three reasons, why a function can be discontinuous: it can **jump**, it can **diverge to infinity**, or it can **oscillate**. An example of a jump appears with $f(x) = \sin(x)/|x|$, a pole $g(x) = 1/x$ leads to a vertical asymptote and the function going to infinity. An example of a function discontinuous due to oscillations is $h(x) = \sin(1/x)$. Its graph is the **devil's comb**.

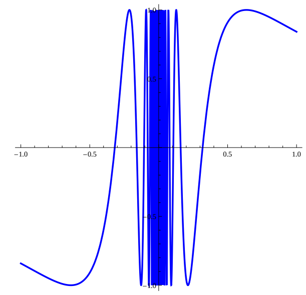
The prototypes in one dimensions are

Jump $f(x) = \sin(x)/|x|$

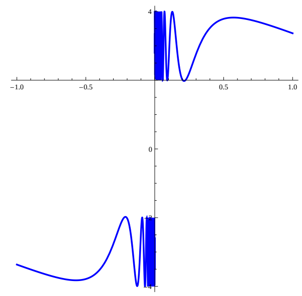


Diverge $g(x) = 1/x$



Oscillate $h(x) = \sin(1/x)$ 

One can have mixtures of these phenomena like the function $3\sin(x)/|x| + \sin(1/x)$, which jumps and also has an oscillatory problem at $x = 0$.



There are two handy tools to discover a discontinuities:
1) Use polar coordinates with coordinate center at the point to analyze the function.
2) Restrict the function to one dimensional curves and check continuity on that curve, where one has a function of one variables.

7 Determine whether the function $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is continuous at $(0,0)$. **Solution** Use polar coordinates to write this as $\sin(r^2)/r^2$ which is continuous at 0 (apply l'Hopital twice if you want to verify this).

8 Is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous at $(0,0)$? **Solution** Use polar coordinates to see that this $\cos(2\theta)$. We see that the value depends on the angle only. Arbitrarily close to $(0,0)$, the function takes any value from -1 to 1.

9 Is the function $f(x,y) = \frac{x^2y}{x^4+y^2}$ continuous? **Solution**. Look on the line $x^2 = y$ to get the function $x^4/(2x^4) = 1/(2x^2)$. It is not continuous at 0. This example is a **real shocker** because it is continuous through each line through the origin: if $y = ax$, then $f(x,ax) = ax^3/(x^4 + a^2x^2) = ax/(x^2 + a^2)$ which is goes to zero for $x \to 0$ as long as $a \neq 0$. If $a = 0$ however, we have $y = 0$ and $f = 0/x^4$ which can be continuously extended to $x = 0$ too.

10 What about the function $f(x,y) = \frac{xy^2+y^3}{x^2+y^2}$? **Solution**. Use polar coordinates and write $r^3\sin^2(\theta)(\cos(\theta) + \sin(\theta)/r^2 = r\sin^2(\theta)(\cos(\theta) + \sin(\theta)$ which shows that the function converges to 0 as $r \to 0$.

11 Is the function $f(x,y) = \frac{\sin(x^2+y^2)}{\sqrt{x^2+y^2}}$ continuous everywhere? **Solution.** Use polar coordinates to see that this is $\sin(r^2)/r^2$. This function is continuous at 0 by Hôpital's theorem.

11: Partial derivatives

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

We also write $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. [1]

1 For $f(x,y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation** for the unknown function $f(x,y)$ involving partial derivatives. The vector $\langle f_x, f_y \rangle$ is called the gradient.

Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$hf_x(x,y) = f(x+h,y) - f(x,y)$
$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x+h,y+h) - (f(x+h,y) - f(x,y))$

$hf_y(x,y) = f(x,y+h) - f(x,y)$.
$h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$

No limits were taken. We established an identity which holds for all $h > 0$, the discrete derivatives f_x, f_y satisfy $f_{xy} = f_{yx}$. It is a "quantum Clairaut" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, the limit $h \to 0$ leads to the classical Clairaut's theorem. The quantum Clairaut theorem holds for **any** functions $f(x,y)$ of two variables. Not even continuity is needed.

2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6y^{10}\cos(y)$. Answer: Do not compute, but think.

3 The continuity assumption for f_{xy} is necessary. The example $f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$ contradicts Clairaut's theorem:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

[1] $\partial_x f, \partial_y f$ were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences".

Here are some examples of partial differential equations. You should know the first 4 well.

4 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $\boxed{f(t,x) = \sin(x-t) + \sin(x+t)}$ satisfies the wave equation.

5 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $\boxed{f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}}$ satisfies the heat equation.

6 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $\boxed{f(x,y) = x^3 - 3xy^2}$ is an example satisfying the Laplace equation.

7 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $\boxed{f(t,x) = e^{-(x+t)^2}}$ satisfy the advection equation.

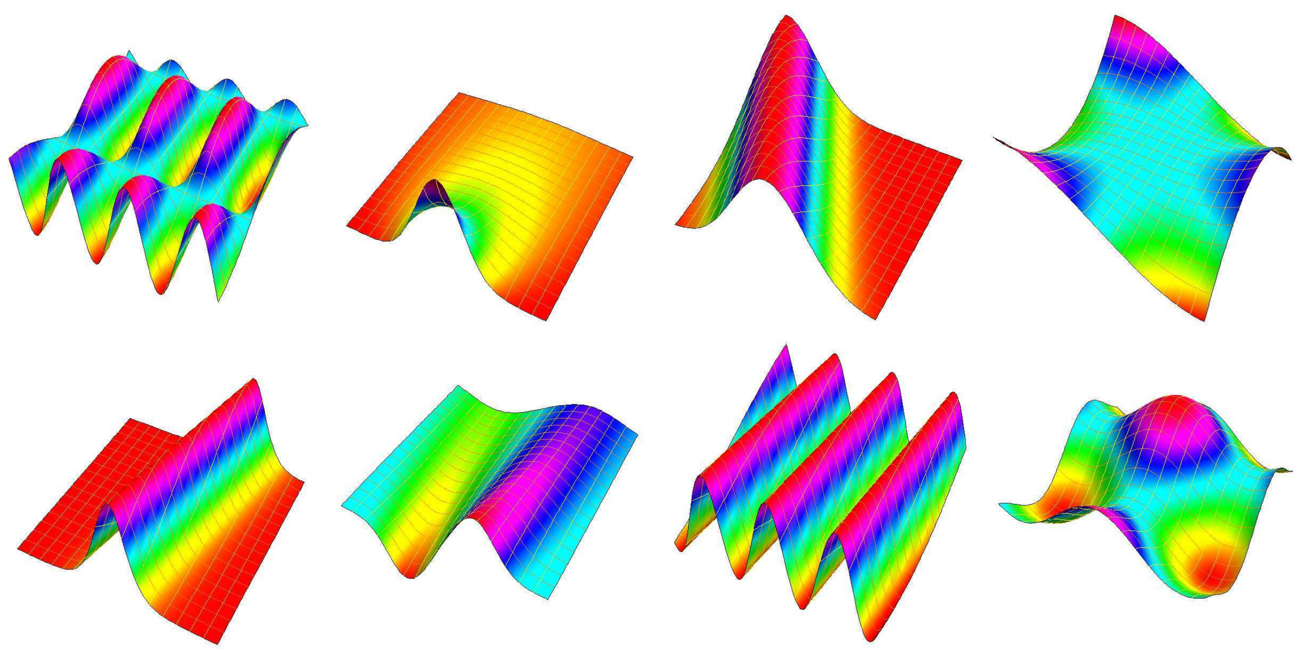
8 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $\boxed{f(x,y) = \sqrt{x^2+y^2}}$ satisfies the eiconal equation.

9 The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $\boxed{f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}}$ satisfies the Burgers equation.

10 The **KdV equation** $f_t + 6 f f_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $\boxed{f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))}$ satisfies the KdV equation.

11 The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $\boxed{f(t,x) = e^{i(kx - \frac{\hbar}{2m} k^2 t)}}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

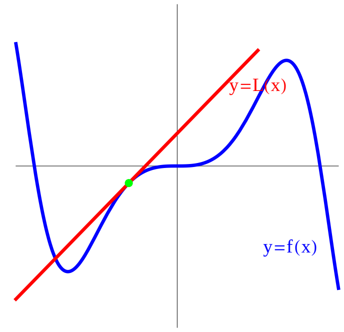
Here are the graphs of the solutions of the equations. Can you match them with the PDE's?



14: Linearization

The **linear approximation** of a function $f(x)$ at a point a is the linear function

$$L(x) = f(a) + f'(a)(x-a) \ .$$



The graph of the function L is close to the graph of f near a. We generalize this to higher dimensions:

The **linear approximation** of $f(x,y)$ at (a,b) is the linear function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \ .$$

The **linear approximation** of a function $f(x,y,z)$ at (a,b,c) is

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) \ .$$

Using the **gradient** $\nabla f(x,y) = \langle f_x, f_y \rangle$ rsp. $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$, the linearization can be written as $L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$. By keeping the second variable $y = b$ is fixed, we get to a one-dimensional situation, where the only variable is x. Now $f(x,b) = f(a,b) + f_x(a,b)(x-a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0,y) = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$. Please avoid the notion of **differentials**. It is a relict from old times.

1 What is the linear approximation of the function $f(x,y) = \sin(\pi x y^2)$ at the point $(1,1)$? We have $(f_x(x,y), y_f(x,y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = \langle \pi \cos(\pi), 2\pi \cos(\pi) \rangle = \langle -\pi, 2\pi \rangle$.

2 Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 \ .$$

3 Find the linear approximation to $f(x,y,z) = xy + yz + zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = \langle y+z, x+z, y+x \rangle, \nabla f(1,1,1) = \langle 2,2,2 \rangle$. we have $L(x,y,z) = f(1,1,1) + \langle 2,2,2 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

4 Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x \sqrt{y} z$.
Solution: take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. **Solution.**The gradient

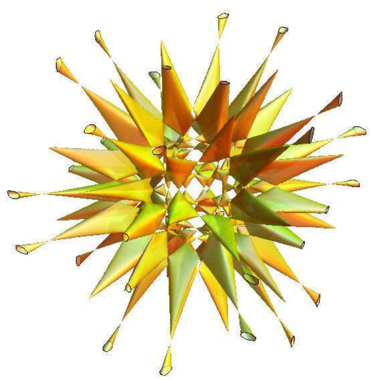
is $\nabla f(x,y,z) = (e^x\sqrt{y}z, e^x z/(2\sqrt{y}), e^x\sqrt{y})$. At the point $(x_0,y_0,z_0) = (0,25,1)$ the gradient is the vector $(5,1/10,5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x-x_0,y-y_0,z-z_0) = 5+(5,1/10,5)(x-0,y-25,z-1) = 5x+y/10+5z-2.5$. We can approximate $f(0.01,24.8,1.02)$ by $5+\langle 5,1/10,5\rangle\cdot\langle 0.01,-0.2,0.02\rangle = 5+0.05-0.02+0.10 = 5.13$. The actual value is $f(0.01,24.8,1.02) = 5.1306$, very close to the estimate.

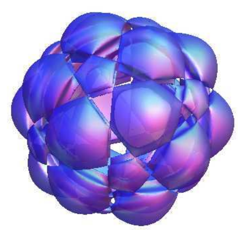
5 Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$. **Solution:** the tangent line is the level curve of the linearlization of $L(x,y)$ of $f(x,y) = y - x^2 = 0$ which passes through the point. We compute the gradient $\langle a,b\rangle = \nabla f(2,4) = \langle -g'(2),1\rangle = \langle -4,1\rangle$ and forming $ax+by = -4x+y = d$, where $d = -4\cdot 2+1\cdot 4 = -4$. The answer is $\boxed{-4x+y=-4}$.

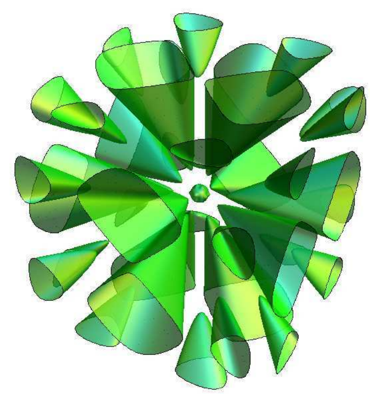
6 The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x,y,z) &= (3+5t)(-1+x^2+y^2+z^2)^2(-2+t+x^2+y^2+z^2)^2 \\ &+ 8(x^2-t^4y^2)(-(t^4x^2)+z^2)(y^2-t^4z^2)(x^4-2x^2y^2+y^4-2x^2z^2-2y^2z^2+z^4)\,, \end{aligned}$$

where $t = (\sqrt{5}+1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5}-1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x,y,z) = 8$. Find the tangent plane of the later surface at the point $(1,1,0)$. **Solution:** We find the level curve of the linearization by computing the gradient $\nabla f(1,1,0) = \langle 64,64,0\rangle$. The surface is $x+y = d$ for some constant d. By plugging in the point $(1,1,0)$ we see that $x+y = 2$.



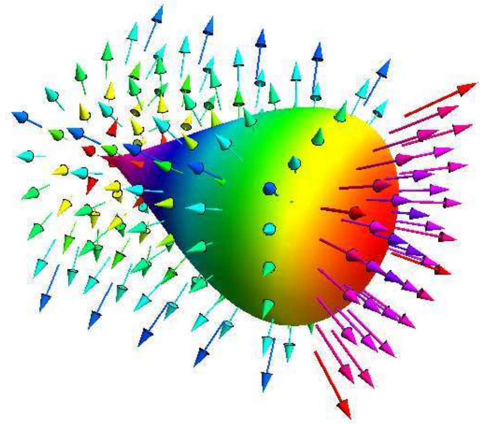




The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

7 is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? **Solution.** We get $\langle a,b,c\rangle = \langle 20,4,4\rangle$ and so the equation of the plane $20x+4y+4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



15: Chain rule

If f and g are functions of one variable t, then the **single variable chain rule** tells

$$\frac{d}{dt} f(g(t)) = f'(g(t))g'(t) \ .$$

For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$. It can be proven by linearizing the functions f and g and verifying the chain rule in the linear case. The **chain rule** is useful to find derivatives like $\arccos'(x)$: write $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))} \arccos'(x)$ $\sqrt{1-x^2} \arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

1 Derive using implicit differentiation the derivative $d/dx \ \arctan(x)$. **Solution.** We have $\sin' = \cos$ and $\cos' = -\sin$ and from $\cos^2(x) + \sin^2(x) = 1$. follows $1 + \tan^2(x) = 1/\cos^2(x)$. Therefore $d/dx \tan(\arctan(x)) = 1/\cos^2(\arctan(x)) \tan'(x) = x$ Now $1/\cos^2(x) = 1/(1 + \tan^2(x)$ so that $\tan'(x) = 1/(1+x^2)$.

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$.

If $\vec{r}(t)$ is curve and f is a function of several variables we can build a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, If $\vec{r}(t)$ is a parametrization of a curve in the plane and f is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

The **multivariable chain rule** is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \ .$$

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \ .$$

Now, the identity

$$\frac{f(x(t+h),y(t+h)) - f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h)) - f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h)) - f(x(t),y(t))}{h}$$

holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t), y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate f with respect to t and y is treated as a constant:

$$\frac{f(\boxed{\text{x(t+h)}}) - f(x(t))}{h} = \frac{[f(\boxed{\text{x(t) + (x(t+h)-x(t))}}) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} \ .$$

Write $H(t) =$ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t) + H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h} .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.

2 We move on a circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ on a table with temperature distribution $f(x, y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x, y) = \langle 2x, -3y^2 \rangle$, $\vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 2\cos(t), -3\sin(t)^2 \rangle \cdot \langle -\sin(t), \cos(t) \rangle = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

From $f(x, y) = 0$ one can express y as a function of x. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain

Implicit differentation: $y' = -f_x/f_y$.

Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x, y, z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x, y)$. We can compute the derivative z_x without actually knowing the function $z(x, y)$. To do so, we consider y a fixed parameter and compute using the chain rule $f_x(x, y, z(x, y))1 + f_z(x, y)z_x(x, y) = 0$. This leads to the following

Implicit differentiation:
$z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ $z_y(x, y) = -f_y(x, y, z)/f_z(x, y, z)$

3 The surface $f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x, y)$ at the point $(x, y, z) = (2, 1, 1)$. **Solution:** $z_x(x, y) = -f_x(2, 1, 1)/f_z(2, 1, 1) = -4/(2/9) = -18$.

4 How does the chain rule relate to other differentiation rules? **Answer.** The chain rule is universal: it implies single variable differentiation rules like the addition, product and quotient rule in one dimensions:
$f(x, y) = x + y, x = u(t), y = v(t), d/dt(x + y) = f_x u' + f_y v' = u' + v'$.
$f(x, y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x, y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

5 Can one prove the chain rule from linearization and just verifying it for linear functions? **Solution**. Yes, as in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x, y) = ax + by - c$ and if the curve $\vec{r}(t) = \langle x_0 + tu, y_0 + tv \rangle$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a, b)$ with $\vec{r}\,'(t) = (u, v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

6 Mechanical systems are determined by the energy function $H(x, y)$, which is a function of two variables. The first variable x is the position and the second variable y is the momentum. The equations of motion for the curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ are called **Hamilton equations**:

$$\begin{aligned} x'(t) &= H_y(x, y) \\ y'(t) &= -H_x(x, y) \end{aligned}$$

In a homework you verify that the energy of a Hamiltonian system is preserved: for every path $\vec{r}(t) = \langle x(t), y(t) \rangle$ solving the system, we have $H(x(t), y(t)) = const$.

16: Gradient and Tangent

The **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$ in three dimensions is an important object. The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. The following theorem is important because it provides a crucial link between calculus and geometry.

Gradient theorem: Gradients are orthogonal to level curves and level surfaces.

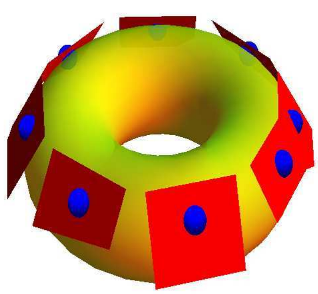
Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$. Because this is true for every curve, the gradient is perpendicular to the surface.

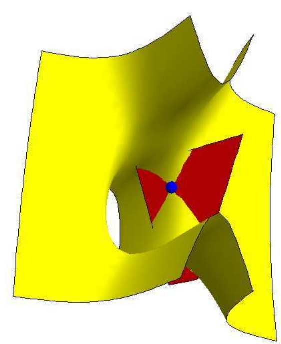
The gradient theorem is useful for example because it allows get tangent planes and tangent lines faster:

The tangent plane through (x_0, y_0, z_0) to a level surface of $f(x,y,z)$ is $ax+by+cz = d$, where $\nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ and d is obtained by plugging in the point.

The statement in two dimensions is completely analog.

1 Find the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z\rangle$. And $\nabla f(1,1,1) = \langle 6,3,2\rangle$. The plane is $6x+3y+2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x+3y+2z = 11$.





2 **Problem:** reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1\rangle$ at the surface $x^4 + y^2 + z^6 = 6$. **Solution:** $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1,-2,1)$. The velocity vector in that ray is $\vec{v} = \langle -1,-1,0\rangle$ The normal vector at this point is $\nabla f(-1,-2,1) = \langle -4,4,6\rangle = \vec{n}$. The reflected vector is

$$R(\vec{v} = 2\text{Proj}_{\vec{n}}(\vec{v}) - \vec{v}\ .$$

We have $\text{Proj}_{\vec{n}}(\vec{v}) = 8/68\langle -4,-4,6\rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4,-4,6\rangle - \langle -1,-1,0\rangle$.

Lecture 16: Tangent spaces

1 Lets compute the tangent line at $(\pi, 0)$ to the curve $y = \sin(x)$ directly by determining the slope and making sure the line goes through the point.

2 Look at $f(x, y) = y - \sin(x) = 0$. Find the gradient $\nabla f(\pi, 0) = \langle a, b\rangle$ of f at $(\pi, 0)$. Now find the tangent line again.

3 Find the tangent plane to the surface $x^2 - y^2 + z^2 = -1$ at the point $(2, 3, 2)$.

4 Find a line perpendicular to the surface $x^2 - y^2 + z^2 = -1$ at the point $(2, 3, 2)$

17: Extrema

An important problem in multi-variable calculus is to **extremize** a function $f(x,y)$ of two variables. As in one dimensions, in order to look for maxima or minima, we consider points, where the "derivative" is zero.

A point (a,b) is called a **critical point** of $f(x,y)$ if $\nabla f(a,b) = \langle 0,0 \rangle$.

Critical points are candidates for extrema because at critical points, all directional derivatives $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ are zero. We can not increase the value of f by moving into any direction.

[1]

1 Find the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x,y) = \langle 4(x^3 - y), 4(y^3 - x) \rangle$ with critical points $(0,0), (1,1), (-1,-1)$.

2 $f(x,y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x,y) = \langle 2x\cos(x^2+y), \cos(x^2+y)+1 \rangle$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at ...$(0,-\pi), (0,\pi), (0,3\pi)$, ...

3 The graph of $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = \langle 2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2) \rangle e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2 + y^2 = 1$. This function has infinitely many critical points.

4 The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = \langle (0,0 \rangle$ for $(x,y) = \dots, (-\pi, 0), (0,0), (\pi,0), (2\pi,0), \dots$. These points are angles for which the pendulum is at rest.

5 The function $f(x,y) = |x| + |y|$ is differentiable on the first quadrant. It does not have critical points there. The function has a minimum at $(0,0)$ but it is not in the domain, where the gradient ∇f is defined.

In one dimension, we needed $f'(x) = 0, f''(x) > 0$ to have a local minimum, $f'(x) = 0, f''(x) < 0$ for a local maximum. If $f'(x) = 0, f''(x) = 0$, then the critical point was undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat inflection point like for $f(x) = x^3$.

Let $f(x,y)$ be a function of two variables with a critical point (a,b). Define $D = f_{xx}f_{yy} - f_{xy}^2$. It is called the **discriminant** of the critical point.

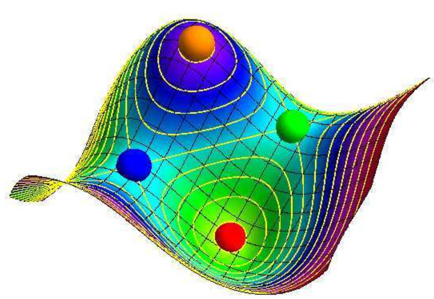
[1]This definition does not include points, where f or its derivative is not defined. We usually assume that functions are nice.

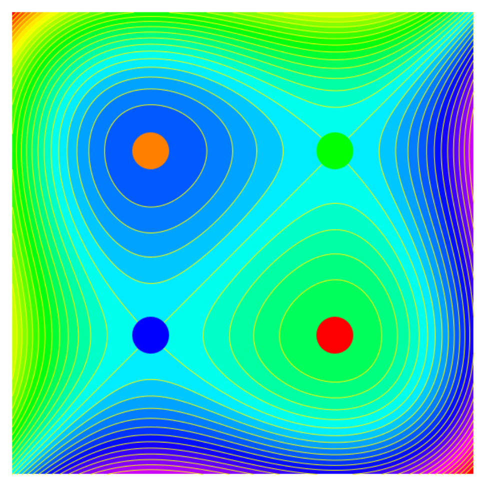
Remark: it can be remembered better if knowing that it is the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

Second derivative test. Assume (a, b) is a critical point for $f(x, y)$.
If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.
If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.
If $D < 0$ then (a, b) is a saddle point.

In the case $D = 0$, we need higher derivatives to determine the nature of the critical point.

6 The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x, y) = \langle x^2 - 1, -y^2 + 1\rangle$. There are 4 critical points $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1, 1)$ we have $D = -4$ and so a saddle point,
For $(-1, 1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1, -1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1, -1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





To determine the maximum or minimum of $f(x, y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see next time how to look for extrema on the boundary.

7 Find the maximum of $f(x, y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x, -1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3, 0)$, $f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

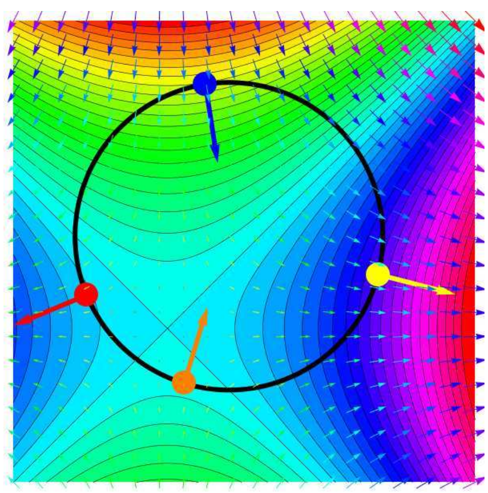
18: Lagrange multipliers

We aim to find maxima and minima of a function $f(x,y)$ in the presence of a **constraint** $g(x,y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel because otherwise the we can move along the curve g and increase f. The directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f or if there is no tangent vector.

The system of equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = 0$ for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Lagrange theorem: Extrema of $f(x,y)$ on the curve $g(x,y) = c$ are either solutions of the Lagrange equations or critical points of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$.



1 Minimize $f(x,y) = x^2 + 2y^2$ under the constraint $g(x,y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

2 Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = \langle 2x, 2y \rangle, \nabla g = \langle 6x^5, 6y \rangle$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1-x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1,0), (-1,0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

3 Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r,h) = h\pi r^2 = 1$. The surface area is $A(r,h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. Calculate $\nabla f(x,y) = (2y, 2x + 4\pi y), \nabla g(x,y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$.

4 On the curve $g(x,y) = x^2 - y^3$ the function $f(x,y) = x$ obviously has a minimum $(0,0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x,y) = 0$.

Remarks.
1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
2) With $g(x,y) = 0$, the Lagrange equations can also be written as $\nabla F(x,y,\lambda) = 0$ where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$.
3) Either of the two properties equated in the Lagrange theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions. The proof is the same:

> Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.

5 Find the extrema of $f(x,y,z) = z$ on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x,y,z) = (0,0,1)$, $\nabla g(x,y,z) = (2x, 2y, 2z)$ and solve $(0,0,1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0,0,-1)$ the maximum the north pole $(0,0,1)$.

6 A dice shows k eyes with probability p_k. Introduce the vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^6 p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + ... + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The fair dice has maximal entropy. Maximal entropy means **least information content**. An unfair dice provides additional information and allows a cheating gambler or casino to gain profit.

7 Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature minimizes the **free energy** $f(p_1, \dots, p_n) = -\sum_i [p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1-\lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The **Gibbs solution** is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$. [1]

[1] This example appears in a book of Rufus Bowen, Lecture Notes in Math, 470, 1978

20: Global Extrema

To determine the maximum or minimum of $f(x,y)$ on a domain, we determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We have to solve both extrema problems with constraints and without constraints.

A point (a,b) is called a **global maximum** of $f(x,y)$ if $f(x,y) \leq f(a,b)$ for all (x,y). For example, the point $(0,0)$ is a global maximum of the function $f(x,y) = 1 - x^2 - y^2$. Similarly, we call (a,b) a **global minimum**, if $f(x,y) \geq f(a,b)$ for all (x,y).

1 Does the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2$ have a global maximum or a global minimum? If yes, find them. **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x,0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x,y) = (x^2-1)^2 + (y-1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

2 Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. **Solution.** With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3,0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3,0), f(4/3,-1)$ shows that $(4/3,0)$ is the global maximum.

3 Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them? **Solution.** The critical points satisfy $\nabla f(x,y) = \langle 0,0 \rangle$ or $\langle 3x^2 - 3, 3y^2 - 12 \rangle = \langle 0,0 \rangle$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

There are no global maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

You can ignore the following Q&A safely. But it might answer some of your questions.
1. **Do global extrema always exist?** Yes, if the region Y is **compact** meaning that for every sequence x_n, y_n we can pick a subsequence which converges in Y. This is equivalent that the domain is **closed and bounded**.

Bolzano's extremal value theorem. Every continuous function on a compact domain has a global maximum and a global minimum.

2. **Why are critical points important?** Critical points are relevant in physics because they represent configurations with lowest energy. Many physical laws describe extrema. The Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$ describing a particle of mass m moving in a field V along a path $\gamma : t \mapsto \vec{r}(t)$ are equivalent to the property that the path extremizes the are length $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$ among all paths.

3. **Why is the second derivative test true?** Assume $f(x,y)$ has the critical point $(0,0)$ and is a quadratic function satisfying $f(0,0) = 0$. Then $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$ with $A = (x + \frac{b}{a}y), B = b^2/a^2$ and discriminant D. You see that if $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x,y) \neq (0,0)$. The point $(0,0)$ is a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x,y) \neq (0,0)$ and the point (x,y) is a local maximum. If $D < 0$, then f takes both negative and positive values near $(0,0)$. For a general function approximate by a quadratic one.

4. **Is there something cool about critical points?** Yes, assume $f(x,y)$ be the height of an island. Assume there are only finitely many critical points and all of them have nonzero determinant. Label each critical point with a $+1$ if it is a maximum or minimum, and with -1 if it is a saddle point. The sum of all these numbers is 1, independent of the island. [1]

5) **Can we avoid Lagrange by substitution?** To extremize $f(x,y)$ under the constraint $g(x,y) = 0$ we find $y = y(x)$ from the second equation and extremize the single variable problem $f(x, y(x))$. To extremize $f(x,y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1-x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution.

6) **Is there a second derivative test for Lagrange?** A second derivative test can be designed using second directional derivative in the direction of the tangent. Instead, we just make a list of critical points and pick the maximum and minimum.

7) **Does Lagrange also work with more constraints?** With two constraints the constraint $g = c, h = d$ defines a curve. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f. Here is a formulation in three dimensions. Extrema of $f(x,y,z)$ under the constraint $g(x,y,z) = c, h(x,y,z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x,y,z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

8) **Why do D and f_{xx} appear in the second derivative test**. They are natural. The discriminant D is a determinant $\det(H)$ of the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. If $D > 0$ then the sign of f_{xx} is the same as the sign of the trace $f_{xx} + f_{yy}$ which is coordinate independent too. The determinant is the product $\lambda_1 \lambda_2$ of the eigenvalues of H and the trace is the sum of the eigenvalues.

9) **What does D mean?** The discriminant D is defined also at points where we have no critical point. The number $K = D/(1 + |\nabla f|^2)^2$ is called the **Gaussian curvature** of the surface. At critical points $K = D$. Curvature is remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. [2]

10) **Is there a 2. derivative test in higher dimensions?** Yes. one can form the second derivative matrix H and look at all the eigenvalues of H. If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. In general eigenvalues have different signs and we have a saddle point type.

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[1] This follows from the **Poincare-Hopf** theorem.

[2] This is the **Theorema Egregia of Gauss**.

21: Double integrals

The integral $\int\int_R f(x,y)\ dxdy$ is defined as the limit of the Riemann sum

$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

when $n \to \infty$.

1 If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\int\int_R f(x,y)\ dxdy$ as the **signed volume** of the solid below the graph of f and above the region R in the $x - y$ plane. The volume below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle, if the function f is continuous: $\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$.

Proof. For every n the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

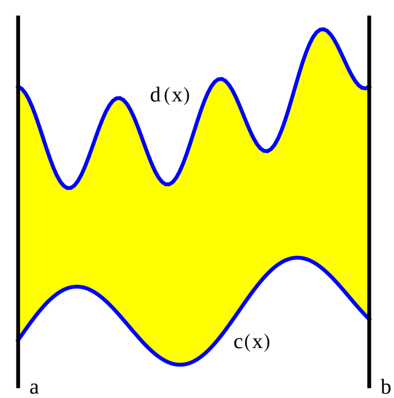
holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.

A **type I region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \}\ .$$

An integral over such a region is called a **type I integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx\ .$$

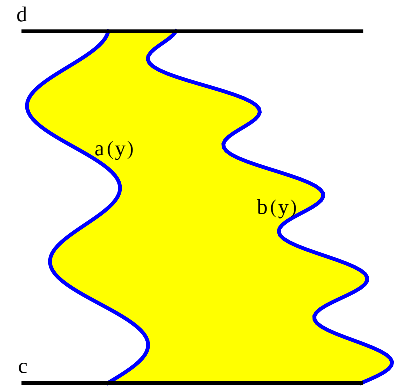


A **type II region** is of the form

$$R = \{(x, y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y)\ dxdy .$$



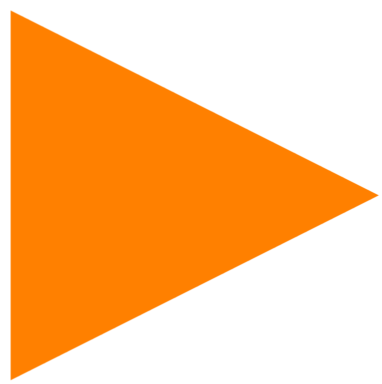
4 Integrate $f(x, y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

This can be solved by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} .$$

5 Integrate $f(x, y) = y^2$ over the region bound by the x-axes, the lines $y = x + 1$ and $y = 1 - x$. The problem is best solved as a type I integral. because we would have to compute 2 different integrals as a type I integral. The y bounds are $x = y - 1$ and $x = 1 - y$



$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4} - \frac{1}{3}) = \frac{1}{10} .$$

6 Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

$$\int\int_R e^{-x^2}\ dxdy\ ?$$

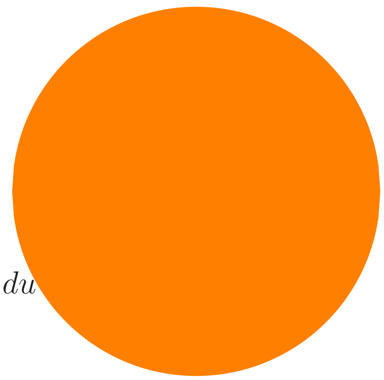
The type II integral $\int_0^1[\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions. The type I integral $\int_0^1[\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$

Lecture 21: Polar integration

1 The area of a disc of radius R is

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1 \; dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2} \; dx \; .$$



This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)} R\cos(u) \; du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u) \; du$$

Using a double angle formula we get $R^2 \int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2} \; du = R^2\pi$. We will now see how to do that better in polar coordinates.

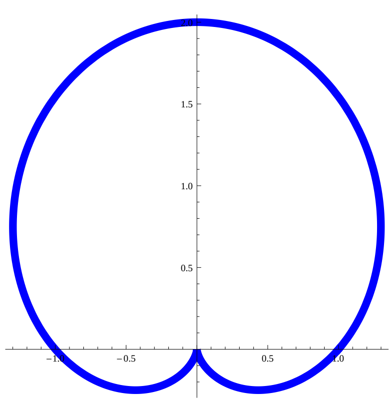
A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

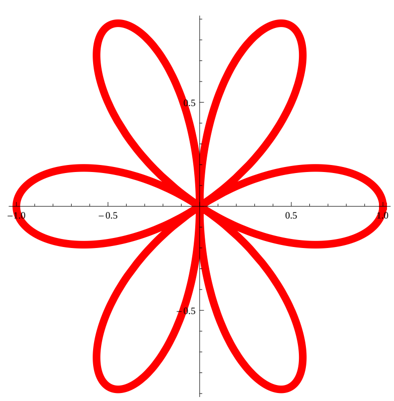
In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = \langle r(t)\cos(\theta(t), r(t)\sin(\theta(t)\rangle$. We are especially interested in regions which are bound by **polar graphs**, where $\theta(t) = t$.

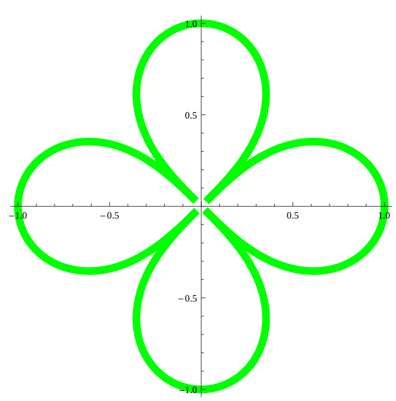
2 The **polar region** defined by $r \leq |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$ they are also called **rhododenea**. These names reflect that polar regions model flowers well.

3 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of a **limacon** a polar curve of the form $r(\theta) = 1 + b\sin(\theta)$.

4 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign. It encloses a flower with two petals.







To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y)\ dxdy = \iint_R f(r\cos(\theta), r\sin(\theta) r\ drd\theta$$

5 Integrate

$$f(x,y) = x^2 + x^2 + xy \ ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi}(r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

6 We have earlier computed area of the disc $\{x^2+y^2 \leq R^2\ \}$ using substitution. It is more elegant to do this integral in polar coordinates: $\int_0^{2\pi}\int_0^R r\ drd\theta = 2\pi r^2/2|_0^R = \pi R^2$.

Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T:(r,\theta)\mapsto(r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\ d\theta dr$.

7 Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi}\int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi}\frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2\ .$$

8 Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2+y^2 < 4, y > 0\ \}$.

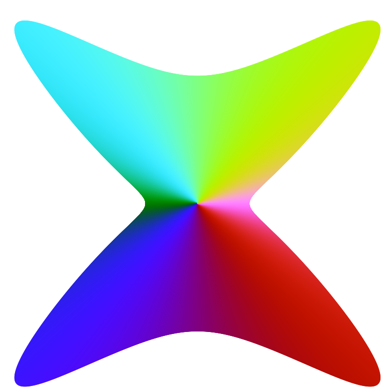
$$\int_1^2\int_0^\pi r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3\int_0^\pi \sin(\theta)\ d\theta dr = \frac{(2^4-1^4)}{4}\int_0^\pi \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms x^2+y^2 try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

9 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b}\right)^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



[1] "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

Lecture 22: Surface area

A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r}+\vec{r}_u]$ and $[\vec{r}, \vec{r}+\vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv$.

1 The parametrized surface $\vec{r}(u,v) = \langle 2u, 3v, 0 \rangle$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = \langle 0,0,6 \rangle$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

For a planar region $\vec{r}(s,t) = P + sv + tw$ where $(s,t) \in G$, the surface area is the area of G times $|v \times w|$.

2 The map $\vec{r}(u,v) = \langle L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v) \rangle$ maps the rectangle $G = [0, 2\pi] \times [0, \pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \; dS = \int_0^{2\pi} \int_0^{\pi} L^2 \sin(v) \; dvdu = 4\pi L^2$

For a sphere of radius L, we have $|\vec{r}_u \times \vec{r}_v| = L^2\sin(v)$ The surface area is $4\pi L^2$.

3 For graphs $(u,v) \mapsto \langle u, v, f(u,v) \rangle$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0, 1, f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1 + f_u^2 + f_v^2} \; dudv$.

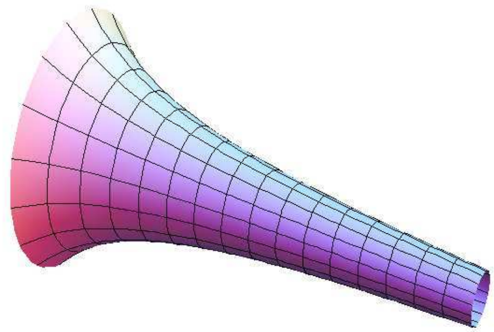
For a graph $z = f(x,y)$ parametrized over G, the surface area is

$$\int\int_G \sqrt{1 + f_x^2 + f_y^2} \; dxdy \; .$$

4 Lets take a surface of revolution $\vec{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u) \rangle$ on $R = [0, 2\pi] \times [a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \; dudv = 2\pi \int_a^b |f(v)|\sqrt{1 + f'(v)^2} \; dv$.

For a surface of revolution $r = f(z)$ with $a \le z \le b$, the surface area is

$$2\pi \int_a^b |f(z)|\sqrt{1+f'(z)^2}\, dz \; .$$



5 Gabriel's trumpet is the surface of revolution where $g(z) = 1/z$, where $1 \le z < \infty$. Its volume is $\int_1^\infty \pi g(z)^2 \; dz = \pi$. We will compute in class the surface area.

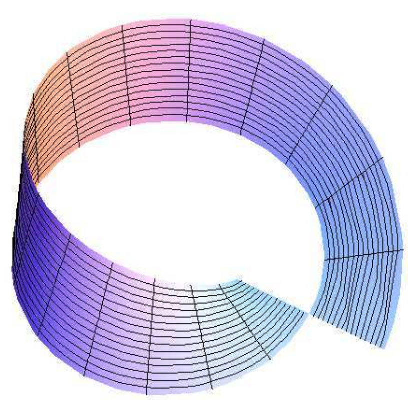
6 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \le 9$. **Solution.** We use polar coordinates in the yz-plane. The paraboloid is parametrized by $(u, v) \mapsto (v^2, v\cos(u), v\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \; dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2} \; dudv = 2\pi \int_0^3 v\sqrt{1+4v^2} \; dv = \pi(37^{3/2}-1)/6$.

7 In this example we derive the distortion factor r in polar coordinates. To do so, we parametrize a region in the xy plane with $\vec{r}(u, v) = \langle u\cos(v), u\sin(v), 0\rangle$. Given a region G in the uv plane like the rectangle $[0, \pi] \times [1, 2]$, we obtain a region S in the xy plane as the image. The factor $|\vec{r}_u \times \vec{r}_v|$ is equal to the radius u. In our example, the surface area is $\int_0^\pi \int_1^2 u \; dudv = \pi(4-1) = 3\pi$. This is the area of the half annulus S. We could have used polar coordinates directly in the xy plane and compute $\int_0^\pi \int_1^2 r \; dr \; d\theta = 3\pi$. But the only thing which has changed are the names of the variables.

The surface parametrized by

$\vec{r}(u, v) = \langle (2+v\cos(u/2))\cos(u), (2+v\cos(u/2))\sin(u), v$

8 on $G = [0, 2\pi] \times [-1, 1]$ is called a **Möbius strip**. What is its surface area? **Solution.** The calculation of $|\vec{r}_u \times \vec{r}_v|^2 = 4 + 3v^2/4 + 4v\cos(u/2) + v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0, 2\pi] \times [-1, 1]$ can only be evaluated numerically, the result is 25.413....



Lecture 24: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n)\in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) \ .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 Assume E is the box $[0,1] \times [0,1] \times [0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1. When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

The two important methods for triple integrals are the "washer method" and the "sandwich method". The washer method from single variable calculus reduces the problem directly to a one dimensional integral. The new sandwich method reduces the problem to a two dimensional integration problem.

The **washer method** slices the solid along the z-axes. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b [\int\int_{R(z)} f(x,y,z)\ dxdy]\ dz$. The **sandwich method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral becomes $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz]\ dxdy$.

2 An important special case of the sandwich method is the volume

$$\int_R \int_0^{f(x,y)} 1\ dzdxdy \ .$$

under the graph of a function $f(x,y)$ and above a region R. It is the integral $\int\int_R f(x,y)\ dA$. What we actually have computed is a triple integral

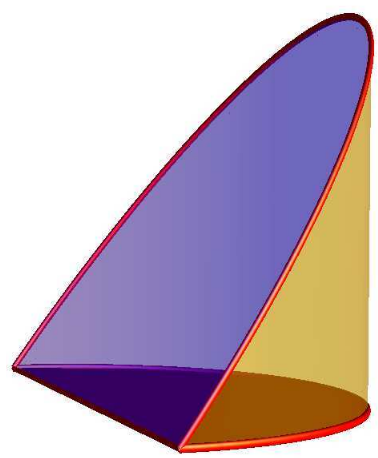
3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA .$$

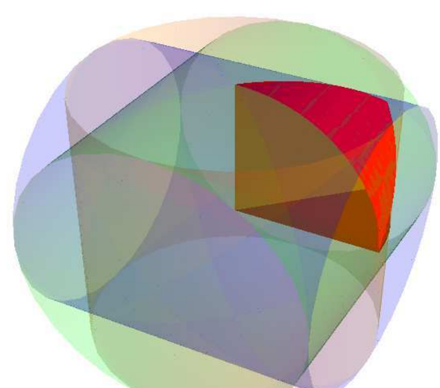
which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2}\, dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2} r\, dr d\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2)\, dz = 4\pi/3$.

4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z)\, dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1. **Solution:**

$$\begin{aligned} \int_0^2 \int_0^6 \int_0^{4-x^2} dz\, dy\, dx &= \int_0^2 \int_0^6 (4-x^2)\, dy dx \\ &= 6\int_0^2 (4-x^2)\, dx = 6(4x - x^3/3)|_0^2 = 32 \end{aligned}$$



5 The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution**. Look from the situation from above and picture it in the $x-y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\, dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos(\theta)\, dr d\theta = 2/3$.



6 Finding the volume of the solid region bound by the three cylinders $x^2+y^2 = 1$, $x^2+z^2 = 1$ and $y^2+z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

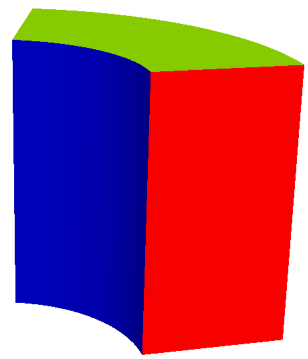
$$16\int_0^{\pi/4}\int_0^1 \sqrt{1-r^2\cos^2(\theta)} r\, dr d\theta$$

has an inner r-integral of $(16/3)(1-\sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1+\sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x)+\sec(x)+\tan(x)$. The result is $16 - 8\sqrt{2}$.

Lecture 25: Spherical integration

Cylindrical coordinates are coordinates in space in which polar coordinates are chosen in the xy plane and the z-coordinate is left untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, produces the same integration factor r as in polar coordinates.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(r,\theta,z)\ \boxed{r}\ drd\theta dz$$

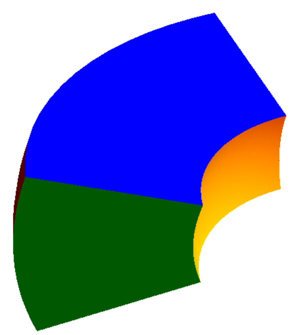


Remember also that spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\ .$$

The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(\rho,\theta,z)\ \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d$$



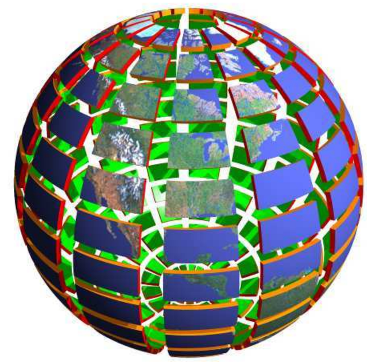
1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin(\phi)\ d\phi d\theta d\rho\ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.

The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2\ dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axis L.

For a sphere of radius R we obtain with respect to the z-axis:



$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \; d\phi d\theta d\rho$$

2

$$= (\int_0^{\pi} \sin^3(\phi) \; d\phi)(\int_0^R \rho^4 \; dr)(\int_0^{2\pi} \; d\theta)$$

$$= (\int_0^{\pi} \sin(\phi)(1 - \cos^2(\phi)) \; d\phi)(\int_0^R \rho^4 \; dr)(\int_0^{2pi} \; d\theta)$$

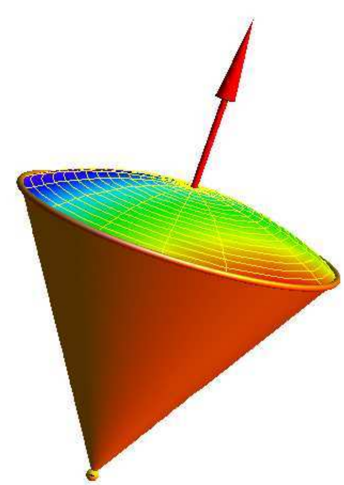
$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^{\pi}(L^5/5)(2\pi) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} \; .$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi) \; d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

Find $\int\int\int_R z^2 \; dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4 \}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is
4 given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have



$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi) \; d\phi d\theta d\rho$$

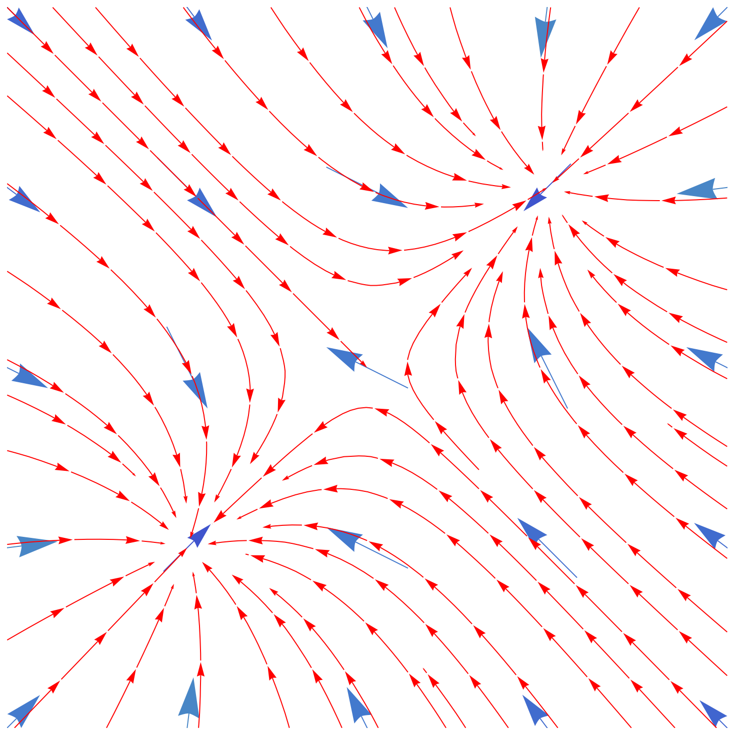
$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2}) \; .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.

Lecture 26: Vector fields

A **vector field** in the plane is a map, which assigns to each point (x, y) a vector $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. A vector field in space is a map, which assigns to (x, y, z) in space a vector $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

For example $\vec{F}(x, y) = \langle x-1, y\rangle/((x-1)^2+y^2)^{3/2} - \langle x+1, y\rangle/((x+1)^2+y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture below:



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, F is no gradient field.

Clairaut test: if $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is not a gradient field.

We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for F to be a gradient field. In class, we see more examples on how to construct the potential f from the gradient field F.

1 Is the vector field $\vec{F}(x, y) = \langle P, Q\rangle = \langle 3x^2y + y + 2, x^3 + x - 1\rangle$ a gradient field? **Solution**: the Clairot test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x, y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^2 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

2 Is the vector field $\vec{F}(x,y) = \langle xy, 2xy^2 \rangle$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

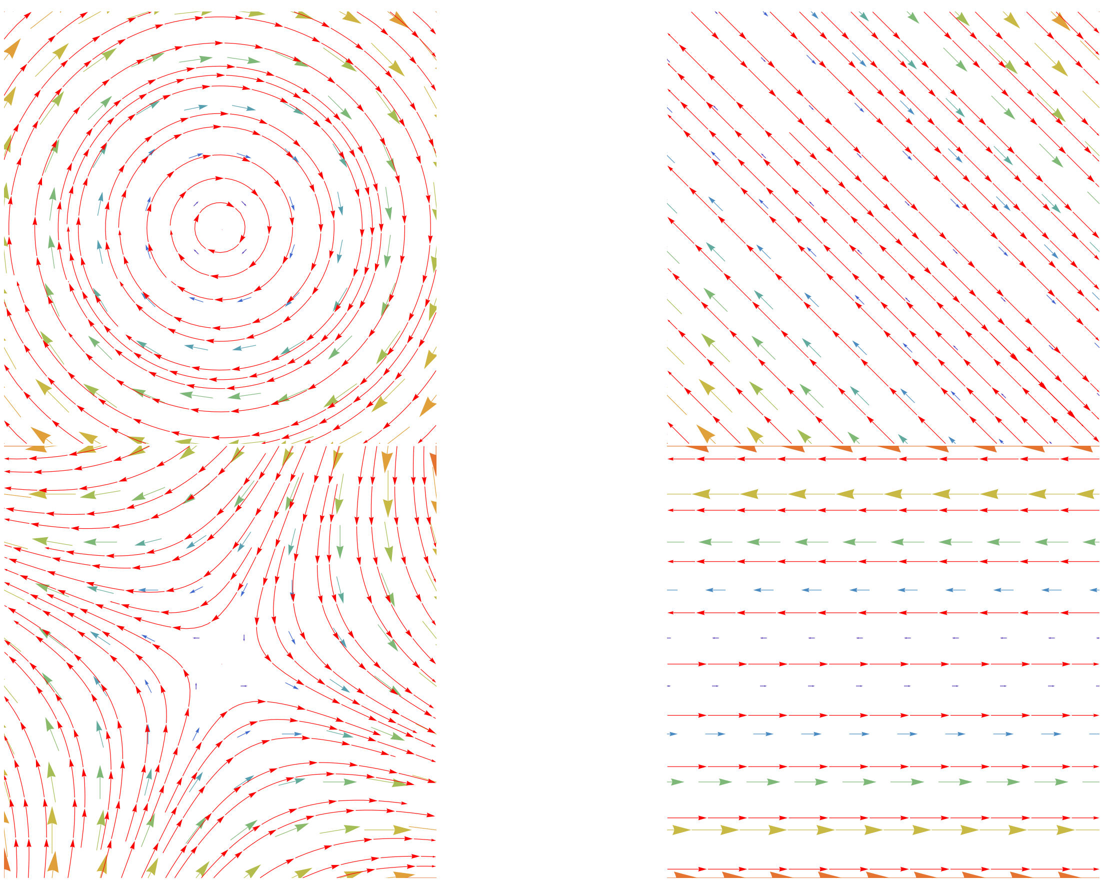
Vector fields are important in differential equations. Motivation comes also from mechanics:

3 A class of vector fields important in mechanics are **Hamiltonian fields**: If $H(x,y)$ is a function of two variables, then $\langle H_y(x,y), -H_x(x,y) \rangle$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x,y) = x^2 + y^2$. Its vector field $(H_y(x,y), -H_x(x,y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H (as you have shown in th homework with the chain rule).

4 Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1,1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x,y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

5 Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$ a gradient field? **No.** Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle \sin(x) + y, \cos(y) + x \rangle$ a gradient field? Yes. the function is $f(x,y) = -\cos(x) + \sin(y) + xy$.

6 Can you spot the following vector fields in the pictures? $F(x,y) = \langle y, 0 \rangle$, $F(x,y) = \langle -y - x, x + y \rangle$, $F(x,y) = \langle -y, x \rangle$, $F(x,y) = \langle y - x, x + y \rangle$. Which ones are conservative?



Lecture 28: Fundamental theorem of line integrals

Recall:

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The following theorem generalizes the fundamental theorem of calculus to higher dimensions:

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

1 Let $\vec{F}(x, y) = \langle 2xy^2 + 3x^2, 2yx^2 \rangle$. Find a potential f of $\vec{F} = \langle P, Q \rangle$.
Solution: The potential function $f(x, y)$ satisfies $f_x(x, y) = 2xy^2 + 3x^2$ and $f_y(x, y) = 2yx^2$. Integrating the second equation gives $f(x, y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x, y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x, y) = x^2y^2 + x^3$. Find g, h from $f(x, y) = \int_0^x P(t, y)\; dt + h(y)$ and $f_y(x, y) = g(x, y)$.

2 At Oliver's last birthday, he relaxed in a Jacuzzi of the Boston Harbor hotel. He moved along curve C which is given by part of the curve $x^{10} + y^{10} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x, y) = \langle x, y^4 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$, the energy you gain from the fluid force.

Lecture 29: Green's theorem

The **curl** of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \; .$$

The curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = \langle P, Q \rangle$ is the scalar $Q_x - P_y$.

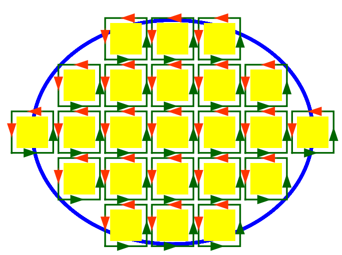
1 For $\vec{F}(x,y) = \langle -y, x \rangle$ we have curl$(F)(x,y) = 2$.

2 If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and curl$(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut's theorem.

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and R is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \int \int_G \text{curl}(F) \; dxdy \; .$$

Proof. The integral of $\vec{F}$ along the boundary of $G = [x, x+\epsilon] \times [y, y+\epsilon]$ is $\int_0^\epsilon P(x+t, y)dt + \int_0^\epsilon Q(x+\epsilon, y+t) \; dt - \int_0^\epsilon P(x+t, y+\epsilon) \; dt - \int_0^\epsilon Q(x, y+t) \; dt$. Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, this is is $(Q_x - P_y)\epsilon^2 \sim \int_0^\epsilon \int_0^\epsilon \text{curl}(F) \; dxdy$. All identities hold in the limit $\epsilon \to 0$.



A general region G can be cut into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the squares is a Riemann sum approximation of the double integral. The boundary integrals converge to the line integral of C.

George Green lived from 1793 to 1841. He was a physicist a self-taught mathematician and miller.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals. and $\int \int_G \text{curl}(F) \cdot dA$ is zero because curl$(F) =$ curl(grad$(f)) = 0$.

The already established Clairaut identity

curl(grad(f)) $= 0$

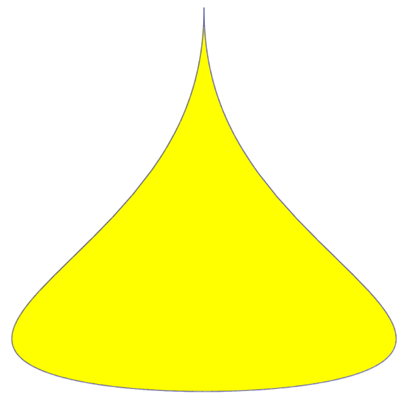
can also remembered as $\nabla \times \nabla f$ noting that the the cross product of two identical vectors is 0. Treating ∇ as a vector is **nabla calculus**.

4 Find the line integral of $\vec{F}(x,y) = \langle x^2-y^2, 2xy\rangle = \langle P, Q\rangle$ along the boundary of the rectangle $[0,2]\times[0,1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y-2y = -4y$ so that $\int_C \vec{F}\, d\vec{r} = \int_0^2 \int_0^1 4y\, dydx = 2y^2|_0^1 x|_0^2 = 4$.

5 Find the area of the region enclosed by

$$\vec{r}(t) = \langle \frac{\sin(\pi t)^2}{t}, t^2 - 1\rangle$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = \langle 0, x\rangle$.



6 An important application of Green is to **compute area**. With the vector fields $\vec{F}(x,y) = \langle P, Q\rangle = \langle -y, 0\rangle$ or $\vec{F}(x,y) = \langle 0, x\rangle$ have vorticity $\text{curl}(\vec{F})(x,y) = 1$. For $\vec{F}(x,y) = \langle 0, x\rangle$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C \langle 0, x(t)\rangle \cdot \langle x'(t), y'(t)\rangle\, dt\ .$$

7 Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $\vec{F}(x,y) = \langle 0, 0\rangle$ there. The line integral is also zero from $(b,0)$ to $(b,f(b))$ and $(a,f(a))$ to $(a,0)$ because $N = 0$. The line integral along the curve $(t,f(t))$ is $-\int_a^b \langle -y(t), 0\rangle \cdot \langle 1, f'(t)\rangle\, dt = \int_a^b f(t)\, dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then curl$(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can be pulled together to a point in R.

If curl$(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\int_C \vec{F}\, d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G, line integrals are path independent and $\vec{F}$ is a gradient field.

Lecture 30: Divergence and Curl

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the vector field

$$\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \ .$$

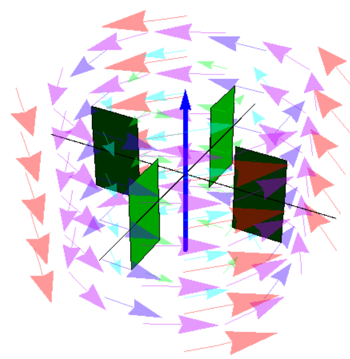
1 The curl of the vector field $\langle x^2 + y^5, z^2, x^2 + z^2 \rangle$ is $\langle -2z, -2x, -5y^4 \rangle$.

We can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. In two dimensions we had

The **curl** of a vector field $\vec{F} = \langle P, Q \rangle$ is $Q_x - P_y$, a scalar field.

If a field has zero curl everywhere, the field is called **irrotational**.

The curl is often visualized using a "paddle wheel". If you place such a wheel into the field into the direction v, its rotation speed of the wheel measures the quantity $\vec{F} \cdot \vec{v}$. We will see why soon. Consequently, the direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl near the vortex center.



In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives, the **gradient**, the **curl** and the **divergence**.

The **divergence** of $\vec{F} = \langle P, Q, R \rangle$ is the scalar field $\text{div}(\langle P, Q, R \rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

The divergence can also be defined in two dimensions, but it is not fundamental.

The **divergence** of $\vec{F} = \langle P, Q \rangle$ is $\text{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

In two dimensions, the divergence is just the curl of a -90 degrees rotated field $\vec{G} = \langle Q, -P \rangle$ because $\text{div}(\vec{G}) = Q_x - P_y = \text{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector" $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$.

$$\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz} \; .$$

is the Laplacian of f. One also writes $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f)$.

From $\nabla \cdot \nabla \times \vec{F} = 0$ and $\nabla \times \nabla \vec{F} = \vec{0}$, we get

$\text{div}(\text{curl}(\vec{F})) = 0$, $\text{curlgrad}(\vec{F}) = \vec{0}$.

2 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = \langle x+y, z, y^2 \rangle = \text{curl}(\vec{G})$?
Answer: No, because $\text{div}(\vec{F}) = 1$ is incompatible with $\text{div}(\text{curl}(\vec{G})) = 0$.

3 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \text{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \text{grad}(f)$. Now, $\text{div}(F) = 0$ implies $\text{divgrad}(f) = \Delta f = 0$.

4 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x,y) = \langle 3x^2 - 3y^2, -6xy \rangle \; .$$

5 If we rotate the vector field $\vec{F} = \langle P, Q \rangle$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = \langle -Q, P \rangle$. The integral $\int_C F \;\cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that $\text{curl}(\vec{F}) = \text{div}(\vec{G})$ and Green's theorem is $\int\int_R \text{div}(\vec{G}) \; dxdy = \int_C \vec{G} \cdot d\vec{n}$. where $dn(x,y)$ is a normal vector at (x,y) orthogonal to the velocity vector $\vec{r}\,'(x,y)$ at (x,y). Do not bother with this, it is just Green's theorem in disguise.

The divergence at a point (x,y) is the flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero

The curl at a point (x,y) is the work done along a small circle of radius r around the point in the limit when the radius of the circle goes to zero

We have now all the derivatives together. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\text{grad}} 1$$

$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$

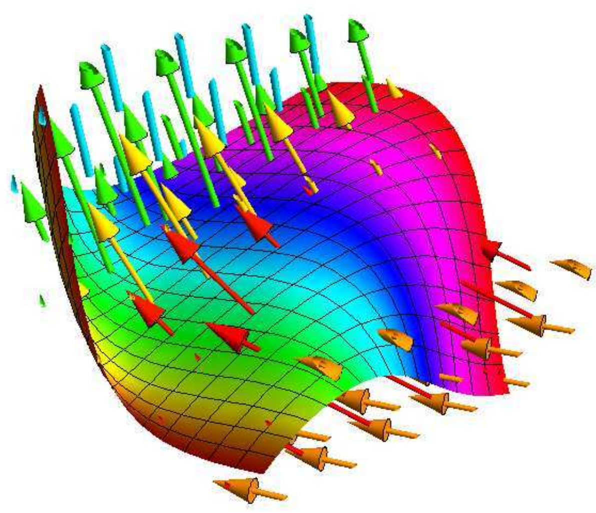
$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

Lecture 31: Flux integrals

If a surface S is parametrized as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain G in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int\int_G \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv\ .$$

With the short hand notation $\vec{dS} = (\vec{r}_u \times \vec{r}_v)\ dudv$ representing an infinitesimal normal vector to the surface, this can be written as $\int\int_S \vec{F} \cdot \vec{dS}$. The interpretation is that if $\vec{F}$ = fluid velocity field, then $\int\int_S \vec{F} \cdot \vec{dS}$ is the amount of fluid passing through S in unit time.
Because $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F} \cdot \vec{n}$ is the normal component of the vector field with respect to the surface. One could write therefore also $\int\int_S \vec{F} \cdot \vec{dS} = \int\int \vec{F} \cdot \vec{n}\ dS$ where dS is the surface element we know from when we computed surface area. The function $\vec{F} \cdot \vec{n}$ is the scalar projection of $\vec{F}$ in the normal direction. Whereas the formula $\int\int 1\ dS$ gave the area of the surface with $dS = |\vec{r}_u \times \vec{r}_v| dudv$, the flux integral weights each area element dS with the normal component of the vector field with $\vec{F}(\vec{r}(u,v) \cdot \vec{n}(\vec{r}(u,v))$. We do not use this formula for computations because computing $\vec{n}$ gives additional work. We just determine the vectors $\vec{F}(\vec{r}(u,v))$ and $\vec{r}_u \times \vec{r}_v$ and integrate its dot product over the domain.



1 Compute the flux of $\vec{F}(x,y,z) = \langle 0, 1, z^2\rangle$ through the upper half sphere S parametrized by

$$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle\ .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u,v)) = \langle 0, 1, \cos^2(v)\rangle$ so that

$$\int_0^{2\pi}\int_0^{\pi} -\langle 0, 1, \cos^2(v)\rangle \cdot \langle \cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)\rangle\ dudv\ .$$

The flux integral is $\int_0^{2\pi}\int_{\pi/2}^{\pi} -\sin^2(v)\sin(u) - \cos^3(v)\sin(v)\ dudv$ which is $-\int_{\pi/2}^{\pi} \cos^3 v \sin(v)\ dv = \cos^4(v)/4|_0^{\pi/2} = -1/4$.

2 Calculate the flux of the vector field $\vec{F}(x,y,z) = \langle 1,2,4z \rangle$ through the paraboloid $z = x^2+y^2$ lying above the region $x^2+y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2 \rangle$ where $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle$ and $\vec{F}(\vec{r}(u,v)) = \langle 1,2,4r^2 \rangle$. We get $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3)\ drd\theta = 2\pi$.

3 Compute the flux of $\vec{F}(x,y,z) = \langle 2,3,1 \rangle$ through the torus parameterized as $\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) \rangle$, where both u and v range from 0 to 2π. **Solution.** There is no computation is needed. Think about what the flux means.

4 Evaluate the flux integral $\int\int_S \langle 0,0,yz \rangle \cdot d\vec{S}$, where S is the surface with parametric equation $x = uv, y = u+v, z = u-v$ on $R: u^2+v^2 \leq 1$. **Solution:** $\vec{r}_u = \langle v,1,1 \rangle, \vec{r}_v = \langle u,1,-1 \rangle$ so that $\vec{r}_u \times \vec{r}_v = \langle -2, u+v, -u+v \rangle$. The flux integral is $\iint_R \langle 0,0,u^2-v^2 \rangle \cdot \langle -2, u+v, -u+v \rangle\ dudv = \iint_R v^2u - u^3 - v^3 + u^2v\ dudv$ which is best evaluated using polar coordinates: $\int_0^1 \int_0^{2\pi} r^4(\sin^2(\theta)\cos(\theta) - \cos^3(\theta) - \sin^3(\theta) + \cos^2(\theta)\sin(\theta))\ d\theta dr = 0$.

5 Evaluate the flux integral $\iint_S \mathrm{curl}(F) \cdot d\vec{S}$ for $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,1] \times [0,1]$ and has an upward orientation. **Solution:** $\mathrm{curl}(F) = \langle -y,-z,-x \rangle$. The parametrization $\vec{r}(u,v) = \langle u, v, 4-u^2-v^2 \rangle$ gives $r_u \times r_v = \langle 2u, 2v, 1 \rangle$ and $\mathrm{curl}(F)(\vec{r}(u,v)) = \langle -v, u^2+v^2-4, -u \rangle$. The flux integral is $\int_0^1 \int_0^1 \langle -2uv + 2v(u^2+v^2-4) - u \rangle\ dvdu = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.

6 What is the relation between the flux of the vector field $\vec{F} = \nabla g / |\nabla g|$ through the surface $S: \{g = 1\}$ with $g(x,y,z) = x^6 + y^4 + 2z^8$ and the surface area of S? **Solution.** The surface area is equal to the flux because $\vec{F} \cdot (\vec{r}_u \times \vec{r}_v) = |\vec{r}_u \times \vec{r}_v|$.

7 Find the flux of the vector field $\vec{G} = \nabla g \times \langle 0,0,1 \rangle$ through the surface S. **Solution.** The flux is zero because the vector field is tangent to the surface.

8 Compute the flux of $\vec{F}(x,y,z) = \langle 2,3,1 \rangle$ through the torus parameterized as $\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) \rangle$, where both u and v range from 0 to 2π. **Solution.** No computation is needed. Think about what the flux means.

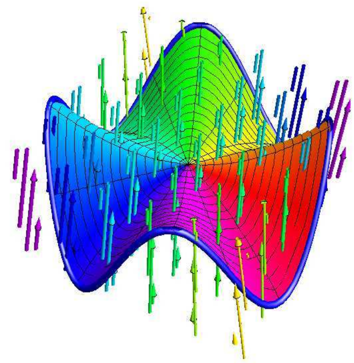
9 You walk in the rain with velocity v along the x axis. The rain can be treated a fluid with velocity $\vec{F} = \langle -v, 0, w \rangle$, where w is the speed of the rain drops. Find the flux through your front and top if we assume you are a rectangular box $[0,1/2] \times [0,1] \times [1/4,1/3]$.
Solution. The flux through the surface S consisting of front and top is the amount of water you soak in per second. Parameterize the front by $\vec{r}(u,v) = \langle 0,u,v \rangle$ in the yz-plane. Your front shirt and trousers catch in unit time the flux of the rain through the front surface. The head gets the rest. We know $\vec{r}_u \times \vec{r}_v = \langle 0,0,1 \rangle$ for the top and $\vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle$ for the front. The flux $\int\int_S \vec{F} \cdot d\vec{S}$ is

$$\int_0^{1/4} \int_0^{1/3} \langle -v,0,w \rangle \cdot \langle 0,0,1 \rangle\ dudv + \int_0^{1/2} \int_0^1 \langle -v,0,w \rangle \cdot \langle 1,0,0 \rangle\ dudv = w/12 + v/2 .$$

If you walk for a fixed distance L and time $T = L/v$, the total flux is $(L/v)(w/12 + v/2) = L/2 + Lw/(2v)$. The part soaked up in the front depends only on L and not on the speeds. The flux caught by head and shoulder is less if you walk fast. So **better run fast!**

Lecture 32: Stokes theorem

The **boundary** of a surface is a curve oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v, a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.



Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \; .$$

1 Let $\vec{F}(x,y,z) = \langle -y, x, 0\rangle$ and let S be the upper semi hemisphere, then $\text{curl}(\vec{F})(x,y,z) = \langle 0,0,2\rangle$. The surface is parameterized by $\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$ on $G = [0,2\pi] \times [0,\pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}(u,v)$ so that $\text{curl}(\vec{F})(x,y,z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi}\int_0^{\pi/2} \sin(2v) \; dvdu = 2\pi$.
The boundary C of S is parameterized by $\vec{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ so that $d\vec{r} = \vec{r}\,'(t)dt = \langle -\sin(t), \cos(t), 0\rangle \; dt$ and $\vec{F}(\vec{r}(t)) \; \vec{r}\,'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π.

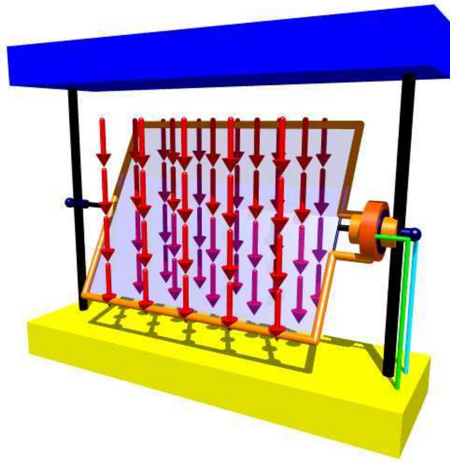
2 If S is a surface in the xy-plane and $\vec{F} = \langle P, Q, 0\rangle$ has zero z component, then $\text{curl}(\vec{F}) = \langle 0,0,Q_x - P_y\rangle$ and $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y \; dxdy$. In this case, Stokes theorem can be seen as a consequence of Green's theorem. The vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F})\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$ is the two dimensional curl: $\vec{F}(\vec{r}(u,v)) \cdot \langle 0,0,1\rangle = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S \vec{F}(\vec{r}(u,v)) \cdot \langle 0,0,1\rangle \; dudv = \int_C \vec{F}(\vec{r}(t)) \; \vec{r}\,'(t)dt$.

3 Calculate the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x, 0\rangle$ through the surface parameterized by $\vec{r}(u,v) = \langle \cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u+\pi/2)\rangle$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

A surface is closed if it bounds a solid.

The flux of the curl of a vector field through a closed surface is zero.

4 The electric field E and the magnetic field B are linked by the **Maxwell equation** $\text{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$. Take a closed wire C which bounds a surface S and consider $\int\int_S B \cdot dS$, the flux of the magnetic field through S. The flux change is related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c\,\text{curl}(\vec{E}) \cdot \vec{dS} = -c\int_C \vec{E}\,\vec{dr} = U$, where U is the voltage measured at the cut-up wire. The flux can be changed by turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. Stokes theorem explains why we can generate electricity from motion.



George Gabriel Stokes

André Marie Ampere

5 Find $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F}(x, y, z) = \langle x^2y, x^3/3, xy\rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above. **Solution.** The curl of F is $\text{curl}(F) = (x, -y, 0)$. We can parametrize the hyperbolic paraboloid as $\vec{r}(u, v) = (u\cos(v), u\sin(v), -u^2\cos(2v))$. $\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos(v), -2u^2\sin(v), u\rangle$. $\vec{F}(\vec{r}(u, v)) = \langle u\cos(v), -u\sin(v), 0\rangle$. $\vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v) = 2u^3$. The integral is $\int_0^1 \int_0^{2\pi} -2r^3\, d\theta dr = \pi$.

6 Evaluate the flux of $\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x\rangle$ through the part S of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards. **Solution.** Stokes theorem assures that the flux integral is equal to the line integral along the boundary of the surface. The boundary is the ellipse $\vec{r}(t) = \langle\cos(t), 2\sin(t)\rangle$, $0 \leq t \leq 2\pi$. The vector field on the xy-plane $z = 0$ is

$$\vec{F}(x, y, 0) = \langle 0, x, ye^{x^2}\rangle \ .$$

To compute the line integral of this vector field along the boundary curve, compute $\vec{r}'(t) = \langle -\sin(t), 2\cos(t), 0\rangle$ and $\vec{F}(\vec{r}(t)) = \langle 0, \cos(t), 2\sin(t)e^{\sin^2(t)}\rangle$. The dot product is the power $2\cos^2(t)$. Integrating this over $[0, 2\pi]$ gives 2π.

Lecture 32: Stokes theorem

To understand Stokes, like with anything in mathematics, you have first to make sure that all definitions are clear. This is already quite tough for Stokes theorem, because it involves a lot of concepts:

- parametrized curves: the boundary of a surface is a curve
- parametrized surfaces: the surface is usually given as such
- line integral $\int_I \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$
- flux integral $\int_G \vec{F}(\vec{r}(u,v)\;\; \cdot (\vec{r}_u \times \vec{r}_v)\; dudv$
- the dot and cross product appear. In the flux integral you can see the triple scalar product
- the result relates a single variable integral with a double integral

Here is the theorem:

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \mathrm{curl}(\vec{F}) \;\; \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \; .$$

Stokes theorem has been assigned as an exam problem in 1854 by George Stokes. George Maxwell, (the one from the Maxwell equations) took the exam. It clearly influenced him. See the exhibit on the course website.

Stokes theorem is important because:

- It is a crucial prototype of a theorem.
- It crowns multivariable calculus reinforcing older concepts. [1]
- When understanding it, you demonstrate ability of abstract thought.
- It explains concepts in physics, especially fluid dynamics and electromagnetism.
- It explains what curl is.

After knowing the definitions which go into the theorem and the statement of the theorem you can use the theorem as a tool. It is at first not so important to worry about the meaning of things too much but to learn how to use the theorem as a tool, for example to solve complicated line integrals or to avoid complex flux integrals. Here are two very typical examples how to use the theorem:

[1] As a general principle, it is always a good idea to learn more about a subject than you actually need.

1 Problem A: Find the flux of the curl of $\vec{F}(x,y,z) = \langle -y+\sin(x)z, x, z^3 \rangle$ through the surface $x^2+y^2/4+z^2+z^4x^2 = 1, z > 0$. Solution. This is a problem where the flux integral would be tough to compute. We would have to parametrize the surface and if we would succeed doing that compute the integral. Much easier is to use Stokes theorem and to compute the line integral of $\vec{F}$ along the boundary of the surface which is an ellipse in the xy plane.

2 Problem B: Find the line integral $\int_C \vec{F}\ d\vec{r}$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where $\vec{F}$ is the vector field

$$\vec{F}(x,y,z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle \ .$$

Use a convenient surface S which has C as a boundary. **Solution** The line integral can not be computed directly. The curl of $\vec{F}$ is $\langle 0, 2x^2+2z^2, 0 \rangle$. It is $\int_0^3 \int_0^{2\pi} 2r^3\ d\theta dr$. The answer is 81π.

Here are two conceptual problems:

3 If S is the surface $x^6+y^6+z^6 = 1$ and assume $\vec{F}$ is a smooth vector field. Explain why $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$. **Solution.** The flux of curl(F) through a closed surface is zero by Stokes theorem and the fact that the surface does not have a boundary.
One can see this also by cutting the surface in two pieces and apply Stokes to both pieces.

4 Assume you look at the surface S which is the boundary of a doughnut. You cut the doughnut in two pieces. Verify that the flux of the curl through one part of the surface is the opposite than the flux through the other. **Solution.** One can understand this with Stokes theorem. The flux of the first part is equal to the line integral along the boundary. The flux through the second is the line integral along the same boundary but in the opposite direction.

5 Why is simply connectedness of the region under consideration needed to verify that if a vector field has zero curl then it is a gradient field? This problem appears in the movie "A beautiful mind".
Solution. Given a closed loop C in space. Simply connectedness means that one can pull together the loop to a point. This pulling together defines a surface S to which the curve C is a boundary. The flux of the curl of F through S is zero. By Stokes theorem, the line integral is zero. Now F has the closed loop property and is therefore a gradient field. If the region is not simply connected like the complement of the z-axes, then we can find vector fields F which have zero curl everywhere in the region but which are not gradient fields. An example is

$$\vec{F}(x,y,z) = \langle -y/(x^2+y^2), x/(x^2+y^2), 0 \rangle \ .$$

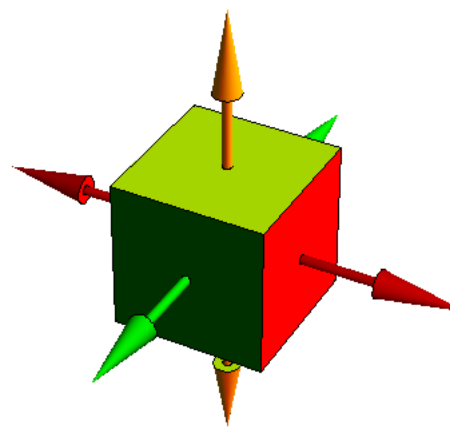
Lecture 33: Divergence theorem

There are three integral theorems in three dimensions. We have seen already the fundamental theorem of line integrals and Stokes theorem. Here is the divergence theorem, which completes the list of integral theorems in three dimensions:

Divergence Theorem. Let E be a solid with boundary surface S oriented so that the normal vector points outside. Let $\vec{F}$ be a vector field. Then

$$\int\int\int_E \mathrm{div}(\vec{F})\ dV = \int\int_S \vec{F}\ \cdot dS\ .$$

To prove this, one can look at a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = \langle P, Q, R \rangle$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot \langle 1, 0, 0 \rangle + \vec{F}(x, y, z) \cdot \langle -1, 0, 0 \rangle] dydz = P(x+dx, y, z) - P(x, y, z) = P_x\ dxdydz$. Similarly, the flux through the y-boundaries is $P_y\ dydxdz$ and the flux through the two z-boundaries is $P_z\ dzdxdy$. The total flux through the faces of the cube is $(P_x + P_y + P_z)\ dxdydz = \mathrm{div}(\vec{F})\ dxdydz$. A general solid can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces. and fluxes through adjacent sides cancel. The sum of all the fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\mathrm{div}(\vec{F})\ dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \mathrm{div}(\vec{F})\ dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain the divergence theorem.



The theorem explains what divergence means. If we average the divergence over a small cube is equal the flux of the field through the boundary of the cube. If this is positive, then more field exists the cube than entering the cube. There is field "generated" inside. The divergence measures the expansion of the field.

1 Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and let S be sphere. The divergence of $\vec{F}$ is the constant function $\mathrm{div}(\vec{F}) = 3$ and $\int\int\int_G \mathrm{div}(\vec{F})\ dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v\ dudv = \int\int_S |\vec{r}(u, v)|^2 \sin(v)\ dudv = \int_0^{\pi} \int_0^{2\pi} \sin(v)\ dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

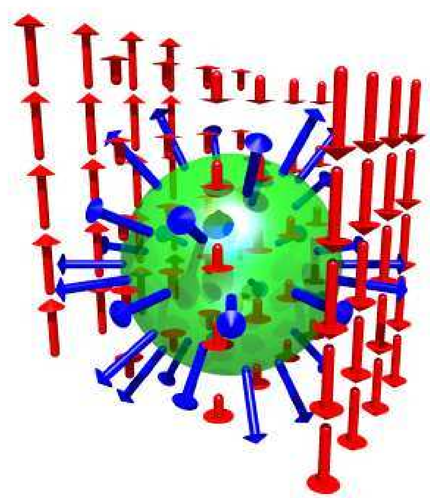
2 What is the flux of the vector field $\vec{F}(x,y,z) = \langle 2x, 3z^2 + y, \sin(x) \rangle$ through the solid $G = [0,3] \times [0,3] \times [0,3] \setminus ([0,3] \times [1,2] \times [1,2] \cup [1,2] \times [0,3] \times [1,2] \cup [0,3] \times [0,3] \times [1,2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: $\operatorname{div}(\vec{F}) = 2$ and so $\int\int\int_G \operatorname{div}(\vec{F})\, dV = 2\int\int\int_G \, dV = 2\mathrm{Vol}(G) = 2(27-7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

3 Find the flux of curl(F) through a torus if $\vec{F} = \langle yz^2, z + \sin(x) + y, \cos(x) \rangle$ and the torus has the parametrization

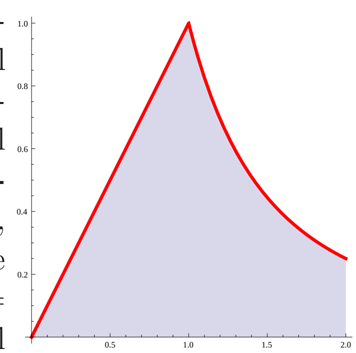
$$\vec{r}(\theta,\phi) = \langle (2+\cos(\phi))\cos(\theta), (2+\cos(\phi))\sin(\theta), \sin(\phi) \rangle \ .$$

Solution: The answer is 0 because the divergence of curl(F) is zero. By the divergence theorem, the flux is zero.

4 Similarly as Green's theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field $\vec{F}(x,y,z) = \langle x, 0, 0 \rangle$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\int\int_{\delta G} \langle x, 0, 0 \rangle \cdot d\vec{S} = \mathrm{Vol}(G)$.



5 How heavy are we, at distance r from the center of the earth? **Solution:** The law of gravity can be formulated as $\operatorname{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\int\int_{S_r} \vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \int\int\int_{B_r} \rho(x)\, dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 \vec{F}(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 \vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3\rho/3$.



Lecture 34: Overview

All integral theorems are incarnations of **the fundamental theorem of multivariable Calculus**

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.

$$\boxed{1}$$

$$\boxed{1} \xrightarrow{\text{FTC}} \boxed{1}$$

$$\boxed{1} \xrightarrow{\text{FTL}} \boxed{2} \xrightarrow{\text{Green}} \boxed{1}$$

$$\boxed{1} \xrightarrow{\text{FTL}} \boxed{3} \xrightarrow{\text{Stokes}} \boxed{3} \xrightarrow{\text{Gauss}} \boxed{1}$$

Fundamental theorem of line integrals: If C is a curve with boundary $\{A, B\}$ and f is a function, then

$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

Remarks.
1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The theorem justifies the name **conservative** for gradient vector fields.
4) The term "potential" was coined by George Green who lived from 1793-1841.

1 Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = \langle \cos(5t), \sin(2t), t^2 \rangle$ from $t = 0$ to $t = 2\pi$.
Solution. $\vec{r}(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}(2\pi) = \langle 1, 0, 4\pi^2 \rangle$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f \ \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

Green's theorem. If R is a region with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_R \operatorname{curl}(\vec{F}) \ dxdy = \int_C \vec{F} \cdot \vec{dr} \ .$$

Remarks.
1) The curve is oriented in such a way that the region is to the left.
2) The boundary of the curve can consist of piecewise smooth pieces.
3) If $C : t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$, the line integral is $\int_a^b \langle P(x(t), y(t)), Q(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle \ dt$.
4) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
5) If $\operatorname{curl}(\vec{F}) = 0$ everywhere in the plane, then the field is a gradient field.
7) $\vec{F}(x, y) = \langle 0, x \rangle$ we get an **area formula**.

2 Find the line integral of the vector field $\vec{F}(x,y) = \langle x^4 + \sin(x) + y, x + y^3 \rangle$ along the path $\vec{r}(t) = \langle \cos(t), 5\sin(t) + \log(1 + \sin(t)) \rangle$, where t runs from $t = 0$ to $t = \pi$.

Solution. $\mathrm{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $\vec{r}(t) = \langle -t, 0 \rangle, -1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = \langle -1, 0 \rangle$. The line integral is $\int_{-1}^{1} \langle t^4 - \sin(t), -t \rangle \cdot \langle -1, 0 \rangle \; dt = -t^5/5|_{-1}^{1} = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int \int_S \mathrm{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r} \; .$$

Remarks.

1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.

2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.

3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).

4) The flux of the curl of $\vec{F}$ only depends on the boundary of S.

5) The flux of the curl through a closed surface like the sphere is zero because the boundary is empty.

3 Compute the line integral of $\vec{F}(x,y,z) = \langle x^3 + xy, y, z \rangle$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.

Solution. The path C bounds a surface $S : \vec{r}(u,v) = \langle u, v, 0 \rangle$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\mathrm{curl}(\vec{F})(x,y,z) = \langle 0, 0, -x \rangle$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \langle 0,0,1 \rangle$ so that $\int \int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 \langle 0, 0, -u \rangle \cdot \langle 0, 0, 1 \rangle \; dudv = \int_0^2 \int_0^1 -u \; dudv = -2$.

Divergence theorem: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then

$$\int \int \int_B \mathrm{div}(\vec{F}) \; dV = \int \int_S \vec{F} \cdot d\vec{S} \; .$$

Remarks.

1) The divergence theorem is also called **Gauss theorem**.

2) It can be helpful to determine the flux of vector fields through surfaces.

3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.

4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

4 Compute the flux of the vector field $\vec{F}(x,y,z) = \langle -x, y, z^2 \rangle$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.

Solution. By Gauss theorem, the flux is equal to the triple integral of $\mathrm{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \; dxdydz = (3-0)(2-(-1))(4-1) = 27$.