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Hypergraphs

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HYPERGRAPHS

Combinatorics of Finite Sets

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Combinatorics of Finite Sets

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1989

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ELSEVIER SCIENCE PUBLISHERS B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

Distributors for the U.S.A. and Canada:

ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
655 Avenue of the Americas
New York, N.Y. 10010, U.S.A.

Library of Congress Cataloging-in-Publication Data

Berge, Claude.
[Hypergraphes. English]
Hypergraphs : combinatorics of finite sets / Claude Berge.
p. cm. -- (North-Holland mathematical library ; v. 45)
Translation of: Hypergraphes.
Bibliography: p.
Includes index.
ISBN 0-444-87489-5
1. Hypergraphs. I. Title. II. Series.
QA166.23.B4813 1989
511'.5--dc20 89-11848
CIP

ISBN: 0 444 87489 5

This volume is a translation of Hypergraphes: Combinatoires des ensembles finis, © Bordas, Paris, France, 1987. It is an updated revision of Part 2 of Graphs and Hypergraphs, © North-Holland, Amsterdam, The Netherlands, 1973.

PRINTED IN THE NETHERLANDS

FOREWORD

For the past forty years, Graph Theory has proved to be an extremely useful tool for solving combinatorial problems, in areas as diverse as Geometry, Algebra, Number Theory, Topology, Operations Research and Optimization. It was thus natural to try and generalise the concept of a graph, in order to attack additional combinatorial problems.

The idea of looking at a family of sets from this standpoint took shape around 1960. In regarding each set as a "generalised edge" and in calling the family itself a "hypergraph", the initial idea was to try to extend certain classical results of Graph Theory such as the theorems of Turán and König. Next, it was noticed that this generalisation often led to simplification; moreover, one single statement, sometimes remarkably simple, could unify several theorems on graphs. It is with this motivation that we have tried in this book to present what has seemed to us to be the most significant work on hypergraphs.

In addition, the theory of hypergraphs is seen to be a very useful tool for the solution of integer optimization problems when the matrix has certain special properties. Thus the reader will come across scheduling problems (Chapter 4), location problems (Chapter 5), etc., which when formulated in terms of hypergraphs, lead to general algorithms. In this way specialists in operations research and mathematical programming have also been kept in mind by emphasizing the applications of the theory.

For pure mathematicians, we have also included several general results on set systems which do not arise from Graph Theory; graphical concepts nevertheless provide an elegant framework for such results, which become easier to visualize.

For students in pure or applied mathematics, we have thought it worthwhile to add at the end of each chapter a collection of related problems. Some are still open but many are straightforward applications of the theory to combinatorial designs, directed graphs, matroids, etc., such consequences being too numerous to include in the text itself.

We wish especially to thank Michel Las Vergnas, and also Dominique de Werra and Dominique de Caen, for their help in the presentation. We express our thanks also to the New York University for permission to include certain chapters of this book which were taught in New York during 1985.

Claude Berge

Note*: The longest proofs, and those which are particularly difficult, are indicated in the text by an asterisk; they can easily be skipped on first reading.*

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INDEX OF DEFINITIONS

List of standard symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of integers ≥ 0
$\mathbb{Z}$	Set of all integers
$\emptyset$	The empty set
$\|A\|$	Cardinality of the set A
$\{x/x \text{ such that} \cdots\}$	Set of x such that ...
$(\forall x)$	For every x
$(\exists x)$	There is an x
$a \in A$	a is an element of the set A
$a \notin A$	a is not an element of the set A
$A \cup B$	Union of A and B
$A \cap B$	Intersection of A and B
$A - B$	A minus B (elements of A not belonging to B)
$A \subset B$	The set A is a subset of set B
$A \not\subset B$	A is not contained in B
$A \times B$	Cartesian product of A by B (set of pairs (a,b) with $a \in A$ and $b \in B$)
$(1) \Rightarrow (2)$	Property (1) implies property (2)
$\binom{p}{q} = \frac{p!}{q!(p-q)!}$	Binomial coefficient "p choose q"
$p \equiv q \ (mod\, k)$	The integer p is congruent to q modulo k
$[\frac{p}{q}]$	Integral part of $\frac{p}{q}$ (largest integer $\leq \frac{p}{q}$)
$[\frac{p}{q}]^*$	Smallest integer $\geq \frac{p}{q}$
$((a^i_j))$	Matrix in which the element in the ith row and jth column is a^i_j
$\det((a^i_j))$	Determinant
$\log p$	Neperian (natural) logarithm

For the notations specific to graphs, see the reference: *Graphs* (C. Berge, *Graphs*, North Holland, 1985).

Chapter 1

General Concepts

1. Dual Hypergraphs

Let $X = \{x_1,x_2,...,x_n\}$ be a finite set. A *hypergraph* on X is a family $H = (E_1,E_2,...,E_m)$ of subsets of X such that

(1) $$E_i \neq \emptyset \quad (i = 1,2,...,m)$$

(2) $$\bigcup_{i=1}^{m} E_i = X.$$

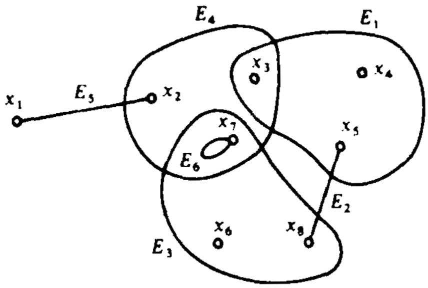
A *simple hypergraph* (or "Sperner family") is a hypergraph $H = (E_1,E_2,...,E_m)$ such that

(3) $$E_i \subset E_j \Rightarrow i = j$$

The elements $x_1,x_2,...,x_n$ of X are called *vertices*, and the sets $E_1,E_2,...,E_m$ are the *edges* of the hypergraph. A simple graph is a simple hypergraph each of whose edges has cardinality 2; a *multigraph* (with loops and multiple edges) is a hypergraph in which each edge has cardinality ≤ 2. Nonetheless we shall not consider isolated points of a graph to be vertices.

A hypergraph H may be drawn as a set of points representing the vertices. The edge E_j is represented by a continuous curve joining the two elements if $|E_j| = 2$, by a loop if $|E_j| = 1$, and by a simple closed curve enclosing the elements if $|E_j| \geq 3$.

One may also define a hypergraph by its incidence matrix $A = ((a_j^i))$, with columns representing the edges $E_1,E_2,...,E_m$ and rows representing the vertices $x_1,x_2,...,x_n$, where $a_j^i = 0$ if $x_i \notin E_j$, $a_j^i = 1$ if $x_i \in E_j$ (cf. Figure 1).



$$
A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array}
\begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

Figure 1. *Representation of a hypergraph H and its incidence matrix*

The *dual* of a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on X is a hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ whose vertices $e_1, e_2, \ldots, e_m$ correspond to the edges of H, and with edges

$$X_i = \{e_j / x_i \in E_j \;\; in \;\; H\}$$

H^* clearly satisfies both conditions (1) and (2).

It is easily seen that the incidence matrix of H^* is the transpose of the incidence matrix of H and so we have $(H^*)^* = H$.

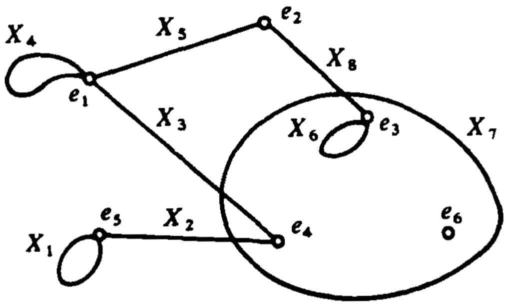


Figure 2. *The dual hypergraph of the hypergraph in Figure 1.*

As for a graph, the *order* of H, denoted by $n(H)$, is the number of vertices. The *number of edges* will be denoted by $m(H)$. Further the *rank* is $r(H) = \max_j |E_j|$, the *anti-rank* is $s(H) = \min_j |E_j|$; if $r(H) = s(H)$ we say that H is a *uniform hypergraph*;

a simple uniform hypergraph of rank r will also be called *r-uniform*, and in this case it will be understood that there is no repeated edge.

For a set $J \subset \{1,2,...,m\}$ we call the family

$$H' = (E_j/j \in J)$$

the *partial hypergraph generated by the set J.* The set of vertices of H' is a nonempty subset of X.

For a set $A \subset X$ we call the family

$$H_A = (E_j \cap A/1 \leq j \leq m,\ E_j \cap A \neq \emptyset)$$

the *sub-hypergraph induced by the set A.* (We define *partial sub-hypergraphs* etc. in a similar fashion).

Proposition. *The dual of a subhypergraph of H is a partial hypergraph of the dual hypergraph H^*.*

In the case of hypergraphs of rank 2 these reduce to the familiar definitions for graphs. All the concepts of graph theory may thus be generalised to hypergraphs which will allow us to find stronger theorems, and applications to objects other than graphs. Further the formulation of a combinatorial problem in terms of hypergraphs sometimes has the advantage of providing a remarkably simple statement having a familiar form.

A stronger result may be much easier to prove than the weak result!

2. Degrees

The other definitions from graph theory which may be extended without ambiguity to a hypergraph H are the following:

For $x \in X$, define the *star* $H(x)$ with centre x to be the partial hypergraph formed by the edges containing x. Define the degree $d_H(x)$ of x to be the number of edges of $H(x)$, so $d_H(x) = m(H(x))$.

The maximum degree of the hypergraph H will always be denoted by

$$\Delta(H) = \max_{x \in X} d_H(x).$$

A hypergraph in which all vertices have the same degree is said to be *regular*.

Note that $\Delta(H) = r(H^*)$, and that the dual of a regular hypergraph is uniform.

For a hypergraph H of order n, the degrees $d_H(x_i) = d_i$ in decreasing order form an n-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ whose properties can be characterised if H is a simple graph (Erdős, Gallai [1960], cf. *Graphs*, Ch. 6, Th. 6). In general

Proposition 1. *An n-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a uniform hypergraph of rank r and order n (possibly with repeated edges) if and only if $\sum_{i=1}^{n} d_i$ is a multiple of r and $d_n \geq 1$.*

Proof. Given such an n-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$, we wish to construct the edges of a hypergraph H one by one on the set $\{x_1, x_2, \ldots, x_n\}$.

In the first step, associate with each vertex x_i a weight $d_i^1 = d_i$ and form the first edge E_1 by taking the r vertices of greatest weight. In the second step, associate with vertex x_i the weight

$$d_i^2 = \begin{cases} d_i^1 & \text{if } x_i \notin E_1 \\ d_i^1 - 1 & \text{if } x_i \in E_1 \end{cases}$$

Form E_2 by taking the r vertices of greatest weight, etc. If $\Sigma d_i = mr$ we obtain H with the edges $E_1, E_2, \ldots, E_m$, and $d_H(x_i) = d_i$ for $i = 1, 2, \ldots, n$.

A hypergraph is *connected* if the intersection graph of the edges is connected. Then we have

Proposition 2 (Tusyadej [1978]). *An n-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a connected uniform hypergraph of rank r if and only if*

(1) $\sum_{i=1}^{n} d_i$ *is a multiple of* r,

(2) $d_i \geq 1 \quad (i = 1, 2, \ldots, n)$,

(3) $\sum_{i=1}^{n} d_i \geq \frac{r(n-1)}{r-1}$,

(4) $$d_1 \leq m = \frac{\Sigma d_i}{r}.$$

(For extensions to non-uniform hypergraphs, cf. Boonyasombat [1984]).

Theorem 1 (Gale [1957], Ryser [1957]). *Given m integers $r_1,r_2,...,r_m$ and an n-tuple of integers $d_1 \geq d_2 \geq \cdots \geq d_n$, there exists a hypergraph $H = (E_1,E_2,...,E_m)$ on a set $X = \{x_1,x_2,...,x_n\}$ such that $d_H(x_i) = d_i$ for $i \leq n$ and $|E_j| = r_j$ for $j \leq m$ if and only if*

(1) $$\sum_{j=1}^{m} \min\{r_j,k\} \geq d_1+d_2+...+d_k \qquad (k < n)$$

(2) $$\sum_{j=1}^{m} r_j = d_1+d_2+...+d_n \ .$$

Proof. We deduce this immediately from the theory of network flows (corollary to theorem 3, Ch.5 in *Graphs*). Indeed, construct a network flow with vertices the points $j = 1,2,...,m$ and $x_1,x_2,...,x_n$, with a source a and a sink z. The arcs are

- all arcs (a,j) with capacity r_j
- all arcs (x_i,z) with capacity d_i
- all arcs (j,x_i) with capacity 1.

It suffices to show that there exists an integer flow satisfying the capacities, saturating each of the arcs (j,z) entering the sink z, that is to say that the maximum flow which can enter set $\{x_i / i \in I\}$ is always greater than or equal to the sum $\sum_{i \in I} d_i$, for all $I \subset \{1,2,...,n\}$. (Further, we note that thanks to the network flow theorem we may always suppose that such a flow never leaves empty an "entry" arc or an "exit" arc.)

Open Problem. *Find a necessary and sufficient condition for an m-tuple (r_j) and an n-tuple (d_i) to be respectively the $|E_j|$ and the $d_H(x_i)$ of a simple hypergraph H.*

Let r,n be integers, $1 \leq r \leq n$. We define the *r-uniform complete hypergraph of order n* (or the *r-complete hypergraph*) to be a hypergraph denoted K_n^r consisting of all the r-subsets of a set X of cardinality n. We may now state in a complete form the celebrated *Sperner's theorem* [1926]; in fact the inequality (1), which allows for a simple proof was discovered (independently) much later by Yamamoto, Meshalkin, Lubell and Bollobás.

Theorem 2 (Sperner [1928]; proof by Yamamoto, Meshalkin, Lubell, Bollobas). *Every simple hypergraph H of order n satisfies*

(1) $$\sum_{E \in H} \binom{n}{|E|}^{-1} \leq 1.$$

Further, the number of edges $m(H)$ satisfies

(2) $$m(H) \leq \binom{n}{[n/2]}$$

For $n = 2h$ even, equality in (2) is attained if and only if H is the hypergraph K_n^h. For $n = 2h-1$ odd, equality in (2) is attained if and only if H is the hypergraph K_n^h or the hypergraph K_n^{h+1}.

Proof. Let X be a finite set of cardinality n. Consider a directed graph G with vertices the subsets of X, and with an arc from $A \subset X$ to $B \subset X$ if $A \subset B$ and $|A| = |B|-1$.

Let $E \in H$, the number of paths in the graph G from the vertex $\emptyset$ to the vertex E is $|E|!$, thus the total number of paths from $\emptyset$ to X is $n! \geq \sum_{E \in H} (|E|)!(n-|E|)!$ (as H is a simple hypergraph, a path passing through E cannot pass through $E' \in H$, $E' \neq E$). We thus deduce inequality (1).

For the second part,

$$\binom{n}{|E|} \leq \binom{n}{[n/2]}.$$

whence

$$1 \geq \sum_{E \in H} \binom{n}{|E|}^{-1} \geq m(H)\binom{n}{[n/2]}^{-1}.$$

We immediately deduce inequality (2).

Let H be a hypergraph satisfying equality in (2). Then for all $E \in H$,

(3) $$\binom{n}{|E|} = \binom{n}{[n/2]}.$$

If $n = 2h$ is even, (3) implies that H is h-uniform, and since $m(H) = \binom{n}{h}$ we have $H = K_n^h$, and the proof is achieved.

If $n = 2h+1$, (3) implies that $h \leq |E| \leq h+1$ for all $E \in H$. Let X_k be the set of vertices in G which represent edges of H with cardinality k; the set $X_h \cup X_{h+1}$ is a stable set of G, and $m(H) = |X_h \cup X_{h+1}|$.

The number of arcs of G leaving X_h is equal to $|X_h|(n-h)$; the number of arcs entering the image ΓX_h of X_h is $|\Gamma X_h|(h+1)$. Thus

$$|\Gamma X_h|(h+1) \geq |X_h|(n-h),$$

or

$$|\Gamma X_h| \geq \frac{2h+1-h}{h+1}|X_h| = |X_h|.$$

If X_h is non-empty and is not the set $\mathcal{P}_h(X)$ of all h-subsets of X, the above inequality is strict (because the bipartite subgraph of G generated by the h-subsets and $(h+1)$-subsets is connected), whence

$$m(H) = |X_h| + |X_{h+1}| \leq |X_h| + |\mathcal{P}_{h+1}(X) - \Gamma X_h| .$$
$$< |X_h| + \binom{n}{h+1} - |X_h| = \binom{n}{h+1}.$$

Thus, equality in (2) is possible only if $X_h = \emptyset$ or $X_h = \mathcal{P}_h(X)$, i.e. if $H = K_n^h$ or K_n^{h+1}.

Q.E.D.

For extensions of Theorem 2 see: Erdös [1945], Kleitman [1968], Meshalkin [1963], Kleitman [1965], Greene, Kleitman [1976], Katona [1966], Hochberg, Hirsch [1970], Erdös, Frankl, Katona [1984].

To generalise graphs without "pendent" vertices, we consider the following class of hypergraphs; a hypergraph H is said to be *separable* if for every vertex x, the intersection of the edges containing x is the singleton $\{x\}$ i.e. if $\bigcap_{E \in H(x)} E = \{x\}$.

Corollary. *If an n-tuple $d_1 \geq d_2 \geq \cdots \geq d_n$ of positive integers is the degree sequence of a separable hypergraph $H = (E_1,...,E_m)$ then*

$$\sum_{i=1}^{n} \binom{m}{d_i}^{-1} \leq 1.$$

Essentially H is separable if and only if its dual H^* is a simple hypergraph, which implies, by Theorem 2,

$$\sum_{i=1}^{n} \binom{m}{|X_i|}^{-1} \leq 1.$$

Q.E.D.

To generalise simple graphs, we say that a hypergraph $H = (E_1, E_2, \ldots, E_m)$ is *linear* if $|E_i \cap E_j| \leq 1$ for $i \neq j$. For example, the hypergraphs of Figures 1,2 are linear.

We have immediately

Proposition 3. *The dual of a linear hypergraph is also linear.*

Indeed, if H is linear, two edges X_i and X_j in H^* cannot intersect in two distinct points e_1, e_2, as then, in H, $E_1 \supset \{x_1, x_2\}$, $E_2 \supset \{x_1, x_2\}$, contradicting $|E_1 \cap E_2| \leq 1$.

Theorem 3. *For every linear hypergraph H of order n, we have*

(1) $$\sum_{E \in H} \binom{|E|}{2} \leq \binom{n}{2}.$$

If in addition, H is r-uniform, then the number of edges satisfies

(2) $$m(H) \leq \frac{n(n-1)}{r(r-1)}.$$

The bound in (2) *is attained if and only if H is a Steiner system $S(2,r,n)$.*

For, the number of pairs x,y which are contained in a same edge of H is

$$\sum_{E \in H} \binom{|E|}{2} \leq \binom{n}{2}$$

whence we have (1). If H is r-uniform, (2) follows.

A Steiner system $S(2,r,n)$ is an r-uniform hypergraph on X, with $|X| = n$, in which every pair of vertices is contained in exactly one edge. A necessary and sufficient condition for the existence of an $S(2,3,n)$ system, due to T.P. Kirkman [1847], is that $n \equiv 1$ or 3 (mod 6).

To exclude some values of r it is easily seen that the following are necessary conditions for the existence of $S(2,r,n)$ systems:

(1) $\binom{n}{2}\binom{r}{2}^{-1}$ is an integer;

(2) $(n-1)(r-1)^{-1}$ is an integer.

These conditions are necessary and sufficient for $r = 3,4$ (Hanani). For $r = 6$ these conditions are sufficient with a single exception: no $S(2,6,21)$ system exists. Wilson [1972] has further shown that if r is a prime power and if n is sufficiently large then (1) and (2) are necessary and sufficient.

For all questions on existence and enumeration of $S(2,r,n)$ systems, see Lindner and Rosa [1980]. We give here a list of $S(2,r,n)$ systems known for small values of r and of n:

$S(2,3,7)$	
$S(2,3,9)$	De Pasquale [1899], Brunel [1901], Cole [1913]
$S(2,4,13)$	De Pasquale [1899], Brunel [1901], Cole [1913]
$S(2,3,15)$	Cole [1917], White [1919], Fischer [1940]
$S(2,4,16)$	Witt [1938]
$S(2,3,19)$	Deherder [1976]
$S(2,3,21)$	Wilson [1974]
$S(2,5,21)$	Witt [1938]
$S(2,3,25)$	Wilson [1974]
$S(2,4,25)$	Brouwer, Rokowska [1977]
$S(2,5,25)$	McInnes [1977]
$S(2,3,27)$	McInnes [1977]
$S(2,4,28)$	Rokowska [1977]

We deduce that the bound in (2) of Theorem 3 is the best possible for $n = 7$, $r = 3$; or for $n = 9$, $r = 3$; etc.

3. Intersecting Families

Given a hypergraph H, we define an *intersecting family* to be a set of edges having non-empty pairwise intersection. For example, for every vertex x of H, the star $H(x) = \{E/E \in H,\ x \in E\}$ is an intersecting family of H. The maximum cardinality of an intersecting family, which we denote $\Delta_0(H)$, thus satisfies

$$\Delta_0(H) \geq \max_{x \in X} |H(x)| = \Delta(H).$$

In a multigraph, the intersecting families are just the stars and the triangles (perhaps with multiple edges).

Theorem 4. *Every hypergraph H of order n with no repeated edge satisfies*

$$\Delta_0(H) \leq 2^{n-1}.$$

Further, every maximal intersecting family of the hypergraph of subsets of an n-set has cardinality 2^{n-1}.

Proof. Let $\mathcal{A}$ be a maximal intersecting family of the hypergraph of subsets of X, where $|X| = n$.

If $B_1 \notin \mathcal{A}$ then there exists in $\mathcal{A}$ a set A_1 disjoint from B_1 (by the maximality of $\mathcal{A}$); thus $X - B_1 \supset A_1$, whence, for every $A \in \mathcal{A}$, $(X-B_1) \cap A \neq \emptyset$. By virtue of the maximality of $\mathcal{A}$, we deduce that $(X-B_1) \in \mathcal{A}$. Conversely, if $(X-B_1) \in \mathcal{A}$, we have $B_1 \notin \mathcal{A}$. Hence $B \rightarrow X-B$ is a bijection between $\mathcal{P}(X) - \mathcal{A}$ and $\mathcal{A}$, whence

$$|\mathcal{A}| = \frac{1}{2} |\mathcal{P}(X)| = 2^{n-1}.$$

Lemma (Greene, Katona, Kleitman [1975], anticipated by Bollobás). *Let $x_1, x_2, \ldots, x_n$ be points in that order on a circle and let $\mathcal{A} = (A_1, A_2, \ldots, A_m)$ be a family of circular intervals of points such that*

(1) $\qquad |A_i| \leq \frac{n}{2}$ *for all* $i \leq m$;

(2) $\qquad A_i \cap A_j \neq \emptyset$ *for all* i, j , $i \neq j$;

(3) $\qquad A_i \not\subset A_j$ *for all* i, j , $i \neq j$.

Then

(4) $$m \leq \min_i |A_i|;$$

(5) $$\sum_{i=1}^{m} |A_i|^{-1} \leq 1.$$

Equality is attained in (5) if and only if $\mathcal{A}$ is a family of circular intervals of cardinality m each having a point in common.

(*) **Proof.** Let A_1 be a set of minimum cardinality in $\mathcal{A}$. From (2), $A_1 \cap A_i \neq \emptyset$ for $i \neq 1$; and from (3), these $A_1 \cap A_i$ are intervals with one and only one of their ends coinciding with an end of A_1. From (3) these intervals $A_1 \cap A_i$ are all different. Thus the number of possible intervals of this form is $\leq 2(|A_1|-1)$. From (1) and (2) two sets $A_1 \cap A_i$ and $A_1 \cap A_j$ with $i \neq j$, $i \neq 1$, $j \neq 1$ cannot constitute a partition of A_1; thus only half of these possible intervals can occur, which gives us $m-1 \leq |A_1|-1$. Thus, for all i, $|A_i| \geq |A_1| \geq m$, so we have (4) and (5).

Finally, equality in (5) implies

$$1 = \sum_{i=1}^{m} \frac{1}{|A_i|} \leq \frac{m}{|A_1|} \leq 1$$

So we have $|A_i| = |A_1| = m$, for $1 \leq i \leq m$. Thus the A_i are intervals of length m whose initial end-points are m successive points on the circle. Conversely if the A_i satisfy (1), (2), (3) and are all intervals of length m, then clearly we have equality in (5).

Theorem 5 (Erdős, Chao-Ko, Rado [1961], proof by Greene, Katona, Kleitman [1976]). *Let H be a simple intersecting hypergraph of order n and of rank $r \leq n/2$; then*

(1) $$\sum_{E \in H} \binom{n-1}{|E|-1}^{-1} \leq 1;$$

(2) $$m(H) \leq \binom{n-1}{r-1}.$$

Further we have equality in (2) when H is a star of K_n^r (and only then if $r < \frac{n}{2}$).

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of H. For any permutation π of $1, 2, \ldots, n$, denote by H_π the set of edges of H which are intervals for the circular sequence $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{\pi(1)}$. For $E \in H$, put

$$\beta(E) = |\{\pi / E \in H_\pi\}|.$$

From the lemma,

$$\sum_{E \in H_\pi} \frac{1}{|E|} \leq 1. \tag{3}$$

We then have

$$\sum_{E \in H} \frac{\beta(E)}{|E|} = \sum_{E \in H} \sum_{\pi / E \in H_\pi} \frac{1}{|E|} = \sum_{\pi} \sum_{E \in H_\pi} \frac{1}{|E|} \leq n! \tag{4}$$

Let E_0 be an edge of H, with cardinality $|E_0| = h$, and let x_0 be an element of E_0. Since E_0 is also an edge of the hypergraph $K_n^h(x_0) = H'$, and since from the lemma we have equality in (3) for H', we have equality in (4) for H', and

$$\frac{\beta(E_0)}{|E_0|} = \frac{1}{m(H')} \sum_{E' \in H'} \frac{\beta(E')}{|E'|} = \frac{n!}{m(H')} = n! \binom{n-1}{|E_0|-1}^{-1}$$

We may thus write, using (4),

$$\sum_{E \in H} \binom{n-1}{|E|-1}^{-1} = \frac{1}{n!} \sum_{E \in H} \frac{\beta(E)}{|E|} \leq \frac{n!}{n!} = 1.$$

Thus we have (1).

Finally, every $E \in H$ satisfies $|E| \leq r \leq \frac{n}{2}$, so

$$m(H) \binom{n-1}{r-1}^{-1} \leq \sum_{E \in H} \binom{n-1}{|E|-1}^{-1} \leq 1,$$

(2) follows.

Q.E.D.

For extensions to Theorem 5 see Schönheim [1968], Hilton and Milner [1967], Hilton [1979], Erdös, Chao-Ko, Rado [1961], Bollobás [1974], Frankl [1975], Frankl [1976].

If no restriction is made on the rank, then by analogous methods we obtain:

Generalisation (Greene, Kleitman, Katona [1976]). *Let H be a simple hypergraph of order n. If H is intersecting, then*

(1) $$\sum_{\substack{E\in H\\ |E|\le \frac{n}{2}}} \binom{n}{|E|-1}^{-1} + \sum_{\substack{E\in H\\ |E|> \frac{n}{2}}} \binom{n}{|E|}^{-1} \le 1.$$

(2) $$m(H) \le \binom{n}{[\frac{n}{2}]+1}.$$

Further, equality is attained in (2) for $H = K_n^{[\frac{n}{2}]+1}$.

Remark. Theorem 5 shows that

$$\Delta_0(K_n^r) = \begin{cases} \binom{n-1}{r-1} & \text{if } r \le \frac{n}{2} \\ \binom{n}{r} & \text{if } r > \frac{n}{2} \end{cases}$$

More precisely, we shall show that in the r-complete hypergraph K_n^r, the maximum intersecting families are: for $r < \frac{n}{2}$, the stars of the form $K_n^r(x)$; for $r = \frac{n}{2}$, the maximal intersecting families; for $r > \frac{n}{2}$, the set of edges of K_n^r.

- For $r < \frac{n}{2}$ the proof of Theorem 5 implies that the only maximum intersecting families are stars.
- For $r = \frac{n}{2}$, let H_0 be a maximal intersecting family of K_{2r}^r: if $E \in H_0$ then $X-E \notin H_0$.
  If $E \notin H_0$ then there exists an edge $E_j \in H_0$ which does not meet E (by maximality of H_0) thus $X-E = E_j \in H_0$. Thus $|H_0| = \frac{1}{2}m(K_{2r}^r)$. Hence all maximal intersecting families have the same cardinality.

Theorem 6 (Bollobás [1965]). *Let $H = (E_1,E_2,...,E_m,F_1,F_2,...,F_m)$ be a hypergraph of order n with $2m$ edges such that $E_i \cap F_j = \emptyset$ if and only if $i = j$. Then*

(1) $$\sum_{j=1}^{m} \binom{|E_j|+|F_j|}{|E_j|}^{-1} \leq 1.$$

Further, we have equality in (1) if for some integers r,s with $r+s = n$, we have

$$(E_1,E_2,...,E_m) = K_n^r; \quad (F_1,F_2,...,F_m) = K_n^s.$$

(*) **Proof.** Inspired by an idea of Katona, we may prove the result as follows. Let X be the vertex set of H, and let Y be the set of pairs (S_j,T_j) with $S_j,T_j \subset X$, $S_j,T_j \neq \emptyset$, $S_j \cap T_j = \emptyset$. Form a graph G on Y as follows: two vertices (S_j,T_j) and (S_k,T_k) are adjacent if $S_j \cap T_k = \emptyset$ or $S_k \cap T_j = \emptyset$. Given a permutation π on X and a set $S \subset X$, denote by $\bar{S}$ the smallest interval of the sequence $\sigma = (\pi(1),\pi(2), \ldots ,\pi(n))$ which contains the set S, and put

$$Y(\pi) = \{(S,T)/(S,T)\in Y;\ \bar{S}\cap\bar{T}=\emptyset;\ \bar{S} \text{ is before } \bar{T} \text{ in } \sigma\}.$$

If the vertices (S_j,T_j) and (S_k,T_k) of $Y(\pi)$ are non-adjacent then $\bar{S}_j \cap \bar{T}_j = \emptyset$, $\bar{S}_k \cap \bar{T}_k = \emptyset$, $\bar{S}_j \cap \bar{T}_k \neq \emptyset$, $\bar{S}_k \cap \bar{T}_j \neq \emptyset$ which is a contradiction. Thus $Y(\pi)$ is a clique of G.

Note that if in a graph G on a set Y we consider P cliques $C_1,C_2,...,C_p$ and a stable set $S \subset Y$ we obtain, by counting in two different ways the number of pairs (y,C_i) with $y \in S$ and $y \in C_i$,

$$\sum_{y\in S} |\{i/y \in C_i\}| = \sum_{i=1}^{p} |C_i \cap S| \leq p$$

As the (E_j,F_j) for $j = 1,...,m$ constitute a stable set of G we have

(2) $$\sum_{j=1}^{m} |\{\pi/y(\pi)\in(E_j,F_j)\}| \leq n!$$

Further, for two disjoint sets $E,F \subset X$

$$|\{\pi/(E,F)\in Y(\pi)\}| = \binom{n}{|E\cup F|}(n-|E\cup F|)!\,|E|!\,|F|! =$$

$$= n!\binom{|E|+|F|}{|E|}^{-1}$$

This equality, together with (2) gives us relation (1) which was what we had to prove.

4. The coloured edge property and Chvátal's Conjecture

Let $H = (E_1,...,E_m)$ be a hypergraph. The *chromatic index* of H is the least number of colours necessary to colour the edges of H such that two intersecting edges are always coloured differently. This number $q(H)$ has been extensively studied for graphs.

If $\Delta_0(H) = k$, then at least k distinct colours are needed to colour the edges of a family of k intersecting edges; thus

$$q(H) \geq \Delta_0(H) \geq \Delta(H).$$

We say that H has the *coloured edge property* if $q(H) = \Delta(H)$, i.e. it is possible to legally colour the edges of H with $\Delta(H)$ colours.

Example 1. Let X be a set of individuals; suppose that certain individuals wish to have meetings during the day, each meeting being defined by a subset E_j of X. We suppose that each individual wishes to attend k meetings. Then we can complete all the reunions in k days if and only if the hypergraph $H = (E_1,E_2,...,E_m)$ has the coloured edge property (each colour of an optimal colouring allows us to define the meetings of a day).

Example 2: Bipartite graphs. Let H be a bipartite multigraph defined by a partition (X_1,X_2) of X and some edges E with $|E\cap X_1| = 1$, $|E\cap X_2| = 1$. A well known theorem of König states that H has the coloured edge property.

Example 3: Graphs. Let G be a simple graph, and let $\hat{G}$ be the multigraph obtained from G by adjoining a loop to each vertex. Vizing's theorem says that $q(G) \leq \Delta(G)+1 = \Delta(\hat{G})$. Then we may colour the edges of $\hat{G}$ with $\Delta(\hat{G})$ colours: this is the coloured edge property.

Example 4: r-complete hypergraphs of order a multiple of r. All complete graphs K_{2p} of order $2p$ even have the coloured edge property; this is an old theorem of Lucas [1892] which he formulated in the following way: a residence of $2p$ girls go for a walk every day in rows of two. Each girl refuses to find herself twice with the same partner. Can you organise the walks for $2p-1$ days? Each of these walks is determined by a colour of the edges of the complete graph K_{2p}. Place the vertices $0,1,\ldots,n-1$ on a circle as in Figure 3, the first colour being determined by the segments of this figure, the others obtained by rotation of the segments about the centre 0. In 1936 in Berlin, a student of Schur, R. Peltesohn, submitted a thesis showing that a school of $3p$ girls can walk every day in rows of 3, that is to say the complete hypergraph K^3_{3p} has the coloured edge property.

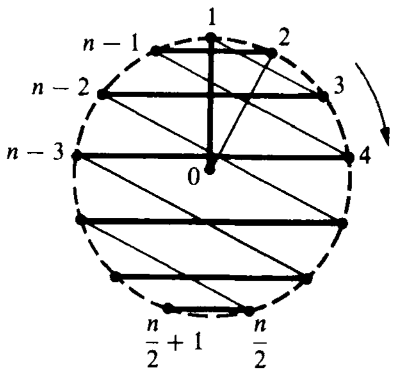


Figure 3

For $p = 3$ this result had been discovered 40 years earlier by Walecki, who had obtained the 28 walks for 9 girls P,Q,a,b,c,d,e,f,g by decomposing the 7 tables shown in Figure 4. Finally, in 1975, Baranyai put a final point on this area of research by showing clearly and simply that K^r_n has the coloured edge property if and only if n is a multiple of r. (For a proof, see §5, Chapter 4).

$$\begin{vmatrix} P & a & b \\ c & d & Q \\ e & f & g \end{vmatrix} \rightarrow \begin{cases} P\,a\,b, & c\,d\,Q, & e\,f\,g \\ P\,c\,e, & a\,d\,f, & b\,Q\,g \\ P\,d\,g, & c\,f\,b, & e\,a\,Q \\ b\,d\,e, & Q\,f\,P, & g\,a\,c \end{cases}$$

$$\begin{vmatrix} P & b & c \\ d & e & Q \\ f & g & a \end{vmatrix}; \begin{vmatrix} P & c & d \\ e & f & Q \\ g & a & b \end{vmatrix}; \begin{vmatrix} P & d & e \\ f & g & Q \\ a & b & c \end{vmatrix}; \begin{vmatrix} P & e & f \\ g & a & Q \\ b & c & d \end{vmatrix}; \begin{vmatrix} P & f & g \\ a & b & Q \\ c & d & e \end{vmatrix}; \begin{vmatrix} P & g & a \\ b & c & Q \\ d & e & f \end{vmatrix}.$$

Figure 4. *The seven tables determining the coloring of the edges of* K_9^3.

Example 5. An *interval hypergraph* is a hypergraph whose vertices are points on a line, and each edge is a set of points in an interval. It is easy to see that such a hypergraph has the coloured edge property. This result is also a special case of a more general theorem which we shall prove in Chapter 5.

Let $H = (E_1, E_2, \ldots, E_m)$ be a simple hypergraph on X: its *hereditary closure* $\hat{H}$ is the hypergraph on X whose edge set is the set of all non-empty subsets $F \subset X$ such that $F \subset E_i$ for at least one index i.

All families $(F_j / j \in J)$ of non-empty subsets of X such that $F \subset F_j \Rightarrow F = F_k$ for some k are called *hereditary*: clearly we may write this in a unique way as the hereditary closure of a simple hypergraph H.

Not all hereditary hypergraphs satisfy the coloured edge property (e.g. $\hat{K}_7^3$, $\hat{K}_9^4$, $\hat{K}_{10}^4$). Nonetheless, in 1974 Chvátal made the important conjecture:

Chvátal's Conjecture. *Every hereditary hypergraph $\hat{H}$ satisfies $\Delta_0(\hat{H}) = \Delta(\hat{H})$.*

In other words, in every hereditary hypergraph there is always a star amongst maximum intersecting families. We shall show various cases of this conjecture.

Theorem 7 (Berge [1976]). *Let H be a star. Then $\hat{H}$ has the coloured edge property.*

Proof. Let H be a simple hypergraph on X, all of whose edges contain a vertex x_0. Assume the theorem to be true for all hypergraphs having fewer than $m(\hat{H})$ edges. Let A be a maximal subset of X of the form $A = E \cup F$ with $E, F \in \hat{H}$. By the maximality of A, we have $x_0 \in A$. Set

$$\mathcal{B} = \{E/E \in \hat{H}, E \cup F = A \text{ for some } F \in \hat{H}\}$$

1. Observe that $\mathcal{B}$ consists of the sets $E_\lambda \in \mathcal{B}$ with $x_0 \in E_\lambda$ and of the sets of the form $A - E_\lambda$; thus we can colour $\mathcal{B}$ with $d_{\mathcal{B}}(x_0)$ colours, using the same colour for $E_\lambda \in \mathcal{B}$ as for $A - E_\lambda$. Thus if $\hat{H} = \mathcal{B}$ we obtain a colouring of H in a number of colours equal to the degree of x_0 in $\mathcal{B}$ and we are done.

2. Suppose $\hat{H} \neq \mathcal{B}$: we shall show that $\hat{H} - \mathcal{B}$ is an hereditary hypergraph.

Let $E \in \hat{H} - \mathcal{B}$ and $E' \subset E$. Since $E' \in \hat{H}$ it suffices to show that $E' \notin \mathcal{B}$. Otherwise, $E' \cup F' = A$ for an $F' \in \hat{H}$. By maximality of A, we have $E \cup F' = A$; thus $E \in \mathcal{B}$, a contradiction.

3. We now show that the maximal edges of $\hat{H} - \mathcal{B}$ contain x_0. For, otherwise there exists some $E \in max(\hat{H} - \mathcal{B})$ with $x_0 \notin E$. Since H is a star, $E \cup \{x_0\} = E_0 \in \hat{H}$. Thus $E_0 \notin \hat{H} - \mathcal{B}$ (by maximality of E); thus $E_0 \in \mathcal{B}$, thus $E_0 \cup F_0 = A$ for some $F_0 \in H$. Thus $E \cup (F_0 \cup \{x_0\}) = A$ and $E \in \mathcal{B}$: contradiction.

4. By the induction hypothesis, the edges of $\hat{H} - \mathcal{B}$ can be coloured with $d_{\hat{H}-\mathcal{B}}(x_0)$ colours so, by using part 1 above, we may write:

$$\Delta(\hat{H}) \leq q(\hat{H}) \leq d_{\hat{H}-\mathcal{B}}(x_0) + d_{\mathcal{B}}(x_0) = d_{\hat{H}}(x_0) \leq \Delta(\hat{H})$$

Thus equality holds throughout. This shows that x_0 is a vertex of maximum degree in $\hat{H}$ and that $q(\hat{H}) = \Delta(\hat{H})$.

Q.E.D.

The colouring of the edges of the hereditary closure of K_n^r is related to a well known problem in Operations Research, the "cutting-stock problem", which was solved by Gilmore and Gomory in [1961]; in this problem we wish to cut, from a stock of rods of length n, k_1 poles of length 1, k_2 of length 2, ...,k_r of length r, and to minimise the total number of rods.

Theorem 8 (Baranyai). *Let $r \leq n$ be integers. $\hat{K}_n^r$ has the coloured edge property if and only if it is possible to solve without waste the cutting stock problem with $k_i = \binom{n}{i}$ for $i = 1,2,...r$, that is to say, there exists an integer solution (x_j^i) to the system*

$$x_j^i \geq 0,$$

x_j^i is the number of i-subsets to colour with j,

$$\sum_{i=1}^{r} ix_j^i = n \qquad (j = 1,2,\ldots)$$

$$\sum_j x_j^i = \binom{n}{i} \qquad (i = 1,2,\ldots,r).$$

It is clear that this condition is necessary; it is also sufficient, as we shall show later (Corollary to Baranyai's Theorem, §5, Chapter 4).

Just as the r-complete hypergraph K_n^r generalises the complete graph, we may generalise the complete bipartite graph by the *r-partite complete hypergraph* $K^r_{n_1,n_2,\ldots,n_r}$ defined as follows: let $X^1,X^2,\ldots,X^r$ be disjoint sets with $|X^i| = n_i$ for $i = 1,2,\ldots,r$.

The vertices are the elements of $X^1 \cup X^2 \cup \cdots \cup X^r$, and the edges are all sets of the form $\{x^1,x^2,\ldots,x^r\}$ with $x^1 \in X^1$, $x^2 \in X^2,\ldots,x^r \in X^r$.

Theorem 9 (Berge, Johnson [1977]). *The r-partite complete hypergraph* $K^r_{n_1,n_2,\ldots,n_r}$ *and its hereditary closure have the coloured edge property.*

Proof.

1. Let $H = K^r_{n_1,n_2,\ldots,n_r}$, with $1 \le n_1 \le n_2 \cdots \le n_r$, $r \ge 2$. We shall show that we can colour the edges of H with $\Delta(H) = n_2n_3\ldots n_r$ colours. We denote the elements of X^k by $x_1^k = 0, x_2^k = 1,\ldots,x_{n_k}^k = n_k-1$. As usual, denote by $[p]_k$ the integer $\le k-1$ congruent to p modulo k. Associate with each edge $\bar{x} = x^1x^2\ldots x^r$ of H the $(r-1)$-tuple

$$\xi(\bar{x}) = ([x^2+x^1]_{n_2},[x^3+x^1]_{n_3},\ldots,[x^r+x^1]_{n_r})$$

If two distinct edges $\bar{x} = x^1x^2\ldots x^r$ and $\bar{y} = y^1y^2\ldots y^r$ intersect, then one of the two following cases occurs:

(i) $x^1 = y^1$ and then there is an $i \ge 2$ with $x^i \ne y^i$, so

$$[x^i+x^1]_{n_i} \ne [y^i+y^1]_{n_i} \text{ and } \xi(\bar{x}) \ne \xi(\bar{y});$$

(ii) $x^1 \ne y^1$ and then there is an $i \ge 2$ with $x^i = y^i$, so

$$[x^i+x^1]_{n_i} \ne [y^i+y^1]_{n_i} \text{ and } \xi(\bar{x}) \ne \xi(\bar{y}).$$

We may consider the map $\bar{x} \to \xi(\bar{x})$ as a colouring of the edges, and the number of distinct colours used is at most $n_2n_3 \cdots n_r = \Delta(K^r_{n_1,n_2,\ldots,n_r})$.

Q.E.D.

2. We shall show that $\hat{H}$ can be coloured with $\Delta(\hat{H})$ colours. For $i = 1,2,...,n$, consider an additional vertex a^i, and put $Y^i = X^i \cup \{a^i\}$; consider the hypergraph $H' = K^r_{n_1+1,n_2+1,...,n_r+1}$ determined by the classes Y^i, $|Y^i| = n_i+1$.

For each edge E of $\hat{H}$ there is an edge F of H' defined by $F = E \cup \{a^i / E \cap X^i = \emptyset\}$.

Thus there is a bijection between the edges of $\hat{H}$ and those of H'. As we have shown that the r-partite complete hypergraph has the coloured edge property, we can colour the edges of H' with

$$\Delta(H') = (n_2+1)(n_3+1)...(n_r+1)$$

colours. If we colour each edge E of $\hat{H}$ with the colour of the corresponding edge F of H', it is clear that two edges of $\hat{H}$ which intersect have different colours. Hence

$$q(\hat{H}) \leq q(H') = \Delta(H') = \Delta(\hat{H}) \leq q(\hat{H}).$$

Thus $q(\hat{H}) = \Delta(\hat{H})$ and the hypergraph $\hat{H}$ has the coloured edge property.

Q.E.D.

The main hypergraphs H for which it has been shown that $\Delta_0(\hat{H}) = \Delta(\hat{H})$ are the following:

1. *H is a star* (Schönheim [1973]). In this case, Theorem 7 shows that $\hat{H}$ has a stronger property, the coloured edge property.

2. *H is 2-uniform* (Vizing).

3. *H is 3-uniform* (Sterboul [1974]). In this case it can also be shown that the maximum intersecting families of $\hat{H}$ have one of the following structures:

- $\hat{H}(a)$ (star);
- $\{ab, ac, bc, abc\}$;
- $\{ab, ac, ad, abc, abd, acd, bcd\}$
- $\{abx_1, abx_2, ...abx_p,\ acx_1, ...acx_p,\ bcx_1, ...bcx_p,\ ab, ac, bc, abc\}$.

4. *H is linear.*

If H is uniform, see Sterboul [1974];

For all H, see Stein [1983].

5. *H is of degree* $\Delta(H) = 2$ (Stein, Schönheim [1978], Wang and Wang [1983]).

6. *H is an r-partite complete hypergraph.* In this case Theorem 9 shows that $\hat{H}$ has the stronger coloured edge property.

7. *H is the complete hypergraph K_n^r* with $r \leq \frac{n}{2}$ (from Theorem 5).

Example 4 suggests the following conjecture:

Conjecture. *If H is linear then $\hat{H}$ has the coloured edge property.*

This conjecture is true if H is a graph (Vizing); if H is a projective plane on 7 points 1,2,..,7, we can colour the edges of $\hat{H}$ with $\Delta(\hat{H}) = 10$ colours in the following way:

colour 1: 123, 45, 6, 7	colour 6: 345, 12, 67
colour 2: 147, 56, 23	colour 7: 367, 14, 25
colour 3: 156, 34, 27	colour 8: 17, 36, 24, 5
colour 4: 246, 37, 15	colour 9: 16, 35, 47, 2
colour 5: 257, 13, 46	colour 10: 57, 26, 1, 3, 4.

5. The Helly property

Let $H = (E_1, E_2, \ldots, E_m)$ be a simple hypergraph. We say that H has the *Helly property* if every intersecting family of H is a star, i.e. for $J \subset \{1,2,\ldots,m\}$,

$$E_j \cap E_k \neq \emptyset \qquad (j,k \in J)$$

implies

$$\bigcap_{j \in J} E_j \neq \emptyset.$$

Hence a graph has the Helly property if and only if its is triangle-free; hypergraphs with the Helly property have also other properties which generalise those of triangle free graphs.

Example 1. Let H be an interval hypergraph: its vertices are points on a line, and its edges are intervals of points. A theorem of Helly shows that H has the Helly property.

Example 2 (Algebra). Let $(X,\leq)$ be a lattice, i.e. an ordered set such that for each pair (a,b) there exists a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$.

Let H be a family of intervals of the form

$$E(a,b) = \{x / a \leq x \leq b\}.$$

Then it can be shown that H has the Helly property. If X is the set of natural numbers, and if the edges of H are arithmetic progressions, the Helly property is known as the "Chinese Remainder Theorem" (cf. Ore [1952]).

We shall say that a hypergraph $H = (E_1, E_2, \ldots, E_m)$ is *k-Helly* if for every set $J \subset \{1,2,\ldots,m\}$, the following two conditions are equivalent:

(D_k) $\qquad I \subset J,\ |I| \leq k$, implies $\bigcap_{i \in I} E_i \neq \emptyset$;

(D) $\qquad \bigcap_{j \in J} E_j \neq \emptyset$

Clearly if J satisfies (D) then it also satisfies (D_k); if H is not k-Helly there are also sets J which satisfy (D_k) but not (D).

Clearly, *a hypergraph is 2-Helly if and only if it satisfies the Helly property.* Note also that if a hypergraph is k-Helly, we have $(D_{k+1}) \Rightarrow (D_k) \Rightarrow (D)$; thus a $(k+1)$-Helly hypergraph is also k-Helly.

Example. Let H be a hypergraph such that if each vertex is a point of $\mathbb{R}^d$ and each edge is the set of points contained in a convex set: an interval hypergraph corresponds to the case $d = 1$. A theorem of Helly states that such a hypergraph in $\mathbb{R}^d$ is $(d+1)$-Helly.

Theorem 10 (Berge, Duchet [1975]). *A hypergraph H is k-Helly if and only if for every set A of vertices with* $|A| = k+1$*, the intersection of the edges* E_j *with* $|E_j \cap A| \geq k$ *is non-empty.*

Proof.

1. Let H be a k-Helly hypergraph on X; let A be a subset of X with $|A| = k+1$. Set

$$J = \{j / |E_j \cap A| \geq k\}.$$

We shall show that $\bigcap_{j \in J} E_j \neq \emptyset$.

Case 1. $|J| \leq k$. We have $\bigcap_{j\in J} E_j \neq \emptyset$ since otherwise the bipartite incidence graph G of the vertices of A versus the edges $(E_j / j \in J)$ satisfies

$$|J|k \leq \sum_{j\in J} d_G(j) = m(G) \leq (|J|-1)|A| = (|J|-1)(k+1)$$

which implies $|J| \geq k+1$: a contradiction.

Case 2. $|J| \geq k+1$. In this case each set $I \subset J$ with $|I| \leq k$ satisfies $\bigcap_{i\in I} E_i \neq \emptyset$ (from Case 1); thus J satisfies (D_k) and hence (D). Thus

$$\bigcap_{j\in J} E_j \neq \emptyset.$$

2. Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph such that for each $A \subset X$ with $|A| = k+1$, the family $(E_j / |E_j \cap A| \geq k)$ has a non-empty intersection. We shall show that H is k-Helly, that is for every $J \subset \{1,2,\ldots,m\}$, $(D_k) \Rightarrow (D)$.

The proof is by induction on $|J|$. Clearly this is true for $|J| \leq k$, so assume $|J| > k$; let $j_1, j_2, \ldots, j_{k+1}$ be distinct elements of J. Then the condition (D_k) implies

$$(\forall I \subset J - \{j_\lambda\}, |I| \leq k): \bigcap_{i\in I} E_j \neq \emptyset$$

By the induction hypothesis this implies

$$\bigcap_{j\in J-\{j_\lambda\}} E_j \neq \emptyset.$$

Let a_λ be an element in this intersection. The elements $a_1, a_2, \ldots, a_{k+1}$ are different (otherwise we are done). For $A = \{a_1, a_2, \ldots, a_{k+1}\}$,

$$|E_j \cap A| \geq k \qquad (j \in J)$$

whence

$$\bigcap_{j\in J} E_j \neq \emptyset.$$

Q.E.D.

Corollary. *A hypergraph H has the Helly property if and only if for any three vertices a_1, a_2, a_3, the family of edges containing at least two of the vertices a_i has a non-empty intersection.*

Application: Family of subtrees of a tree. Let G be an acyclic connected graph on X, i.e. G is a tree. Consider a family H of subsets of X which induce a subtree of

G. We shall show, with the help of the preceding corollary, that H has the Helly property. To see this, consider three vertices a, b, c of G. If $\mu[x,y]$ denotes the unique path in the tree G connecting the vertices x and y it is easy to see that the three paths $\mu[a,b]$, $\mu[b,c]$ and $\mu[c,a]$ have a common vertex x_0 (otherwise G would have a cycle). This vertex x_0 belongs to every edge of H containing two of the points a,b,c. Thus H has the Helly property.

(Note that if G is a path P_n we obtain Helly's Theorem).

Theorem 11 (Tuza [1984]). *Let $H = (E_1,E_2,...,E_m)$ be a simple k-Helly hypergraph of order n. If $\min_j |E_j| \geq k+1$ then*

$$\sum_{j=1}^{m} \binom{n-1}{|E_j|-1}^{-1} \leq 1$$

(*) Proof.

1. We shall show first that every edge E_j contains a vertex a_j such that $E_j - \{a_j\}$ is not contained in any edge other than E_j. Indeed, if this is not the case, there exists an edge $E_0 = \{a_1,a_2,...,a_r\}$ with $r \geq k+1$, such that, say, $E_0 - \{a_i\} \subset E_i$ for $i = 1,2,...,r$. Since H is a simple hypergraph, we have $E_0 \cap E_i = E_0 - \{a_i\}$ for $i = 1,2,...,r$. Thus

$$\bigcap_{j=0}^{r} E_j = \emptyset.$$

However, the intersection of $r-1$ of the sets $E_0,E_1,...,E_r$ is non-empty. Since $r-1 \geq k$, and since H is k-Helly, we have also:

$$\bigcap_{j=1}^{r} E_j \neq \emptyset.$$

A contradiction follows.

2. Thus every edge E_j contains a vertex a_j such that

$$(E_j - \{a_j\}) \cap (X - E_i) \neq \emptyset \qquad \text{for all} \quad i \neq j$$

Set

$$E'_j = E_j - \{a_j\}$$
$$F'_j = X - E_j.$$

Thus we have

$$E'_j \cap F'_j = \emptyset$$

$$E'_j \cap F'_i \neq \emptyset \text{ if } i \neq j$$

We may now apply Theorem 6 and

$$\sum_{j=1}^{m} \binom{|X-E_j|+|E_j|-1}{|E_j|-1} \leq 1,$$

The theorem follows.

Corollary (Bollobás, Duchet [1979]). *Let H be a simple k-Helly hypergraph of order n with $\min_j |E_j| \geq k+2$ and $\max_j |E_j| = r \leq \frac{n}{2}$. Then*

(1) $$m(H) \leq \binom{n-1}{r-1}.$$

Proof. Every $E \in H$ satisfies:

$$\binom{n-1}{|E|-1} \leq \binom{n-1}{r-1}.$$

Hence

$$m(H)\binom{n-1}{r-1}^{-1} \leq \sum_{E \in H} \binom{n-1}{|E|-1}^{-1} \leq 1.$$

Inequality (1) follows.

For a hypergraph H with the Helly property, more precise results can be proved:

Theorem 13 (Bollobás, Duchet [1983]). *Let H be a simple hypergraph of rank $r \geq 3$, $r \leq \frac{n}{2}$, with the Helly property. Then*

(1) $$m(H) \leq \binom{n-1}{r-1}.$$

Further, equality holds in (1) if and only if H is a star of K_n^r.

Theorem 14 (Bollobás, Duchet [1983]). *Let H be a simple hypergraph of order $n \geq 5$ with the Helly property. Then*

(1′) $$m(H) \leq \binom{n-1}{[n/2]}.$$

Further, equality holds in (1′) *if and only if one of the following is true*:

(i) $n = 2h$ *is even and* H *is a star of* K_n^h;

(ii) $n = 2h+1$ *is odd* ≥ 7 *and* H *is a star of* K_n^{h+1};

(iii) $n = 5$ *and* H *is a star of* K_5^3, *or is the bipartite complete graph* $K_{2,3}$ *with one class of* 2 *vertices and one of* 3 *vertices.*

6. Section of a hypergraph and the Kruskal-Katona Theorem

Let H be a simple hypergraph on X of rank r, and let $k \leq r$ be a positive integer. Define the *k-section* of H to be a hypergraph $[H]_k$ whose edges are the sets $F \subset X$ satisfying either $|F| = k$, and $F \subseteq E$ for some $E \in H$; or $|F| < k$ and $F = E$ for some $E \in H$.

Observe that $[H]_k$ is a simple hypergraph on X. Further its rank is k.

For $k = 2$, the 2-section $[H]_2$ is thus a graph; if H contains no loops then $[H]_2$ is a simple graph which is obtained by joining two vertices of X if they belong to the same edge of H. If H is a simple r-uniform hypergraph with m edges, what can we say about the number of edges of $[H]_{r-1}$?

The best possible lower bounds for all m were obtained independently by Kruskal [1963] and Katona [1968]. The proof was simplified by Daykin [1976], and that which we now give, shorter still, is due to Frankl [1984]. We need two preliminary lemmas.

Lemma 1. *Let* m *and* r *be positive integers. Then there exist integers* $a_r, a_{r-1}, \ldots, a_s$ *such that*

(1) $$m = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_s}{s}$$

(2) $$a_r > a_{r-1} > \cdots > a_s \geq s \geq 1.$$

Further the a_i*'s are defined uniquely by* (1) *and* (2). *In particular,* a_r *is the largest integer such that*

$$m - \binom{a_r}{r} \geq 0.$$

(*) **Proof** (by induction on r). For $r = 1$, we have $1 = r \geq s \geq 1$ so $s = 1$ and $a_s = m$; thus the decomposition (1) exists and is unique. Assume now the existence and uniqueness of decomposition (1) for $r-1$. Let a_r be the largest integer such that $m - \binom{a_r}{r} \geq 0$. Then by the induction hypothesis,

$$m - \binom{a_r}{r} = \binom{a_{r-1}}{r-1} + \cdots + \binom{a_s}{s}$$

$$a_{r-1} > a_{r-2} > \cdots > a_s \geq s.$$

We must have $a_r > a_{r-1}$ since otherwise we could write

$$m \geq \binom{a_r}{r} + \binom{a_{r-1}}{r-1} \geq \binom{a_r}{r} + \binom{a_r}{r-1} = \binom{a_r+1}{r}.$$

This contradicts the definition of a_r. Hence the existence of decomposition (1) is proven.

To show uniqueness, suppose there exist two distinct decompositions of m:

$$m = \binom{a_r}{r} + \cdots + \binom{a_s}{s} = \binom{b_r}{r} + \cdots + \binom{b_s}{s}$$

Observe that

$$m \leq \binom{a_r}{r} + \binom{a_r-1}{r-1} + \cdots \binom{a_r-r-1}{r} = \binom{a_r+1}{r}$$

If $a_r < b_r$ then

$$m \leq \binom{a_r+1}{r} \leq \binom{b_r}{r} \leq m$$

This implies $m = \binom{a_r+1}{r}$ contradicting the definition of r.

Hence $a_r = b_r$, and as the decomposition of $m - \binom{a_r}{r} = m - \binom{b_r}{r}$ is unique (induction hypothesis) the two decompositions of m are identical.

Lemma 2 (Frankl [1984]). *Let H be an r-uniform hypergraph on X and let $x_1 \in X$. There exists an r-uniform hypergraph H' on X with $m(H') = m(H)$, $m([H']_{r-1}) \leq m([H]_{r-1})$ and satisfying*

$$F \in [H'-H'(x_1)]_{r-1} \quad \Rightarrow \quad F \cup \{x_1\} \in H'.$$

(*) **Proof.** For a vertex $x \neq x_1$, put

$$\sigma_x E = \begin{cases} (E-\{x\}) \cup \{x_1\} & \text{if } x \in E,\ x_1 \notin E \\ E & \text{otherwise} \end{cases}$$

Put $\sigma_x H = \{\sigma_x E/E \in H\}$. It is easy to see that $[\sigma_x H]_{r-1} \subset \sigma_x [H]_{r-1}$. By repeating the operation σ_y on $\sigma_x H$ as many times as necessary we get a hypergraph H' with $m(H') = m(H)$, $m([H']_{r-1} \leq m([H]_r)$, and $\sigma_x H' = H'$ for all $x \neq x_1$.

Theorem 14 (Kruskal, Katona). *Let H be an r-uniform hypergraph with*

$$m(H) = m = \binom{a_r}{r} + \binom{a_r-1}{r-1} + \cdots + \binom{a_s}{s}$$

$$a_r > a_{r-1} > \cdots > a_s \geq s \geq 1.$$

Then

$$m([H]_{r-1}) \geq \binom{a_r}{r-1} + \binom{a_{r-1}}{r-2} + \cdots + \binom{a_s}{s-1}$$

(*) **Proof** (by induction on r *and* m*).*

1. We may assume that H satisfies

(1) $$F \in [H-H(x_1)]_{r-1} \quad \Rightarrow \quad F \cup \{x_1\} \in H$$

(simply by replacing H by the hypergraph H' defined in Lemma 2). Set $H_1 = (E-\{x_1\}/E \in H(x_1))$. Then

(2) $$m([H]_{r-1}) \geq m(H_1) + m([H_1]_{r-2}).$$

2. The theorem holds trivially for $r = 1$ or $m = 1$; proceed now by induction on r and on m.

Suppose first

(3) $$m(H_1) \geq \binom{a_{r-1}}{r-1} + \cdots + \binom{a_s-1}{s-1}$$

By applying the induction hypothesis to the hypergraph H_1 (less some edges if the inequality is strict), we obtain

$$m([H_1]_{r-2}) \geq \binom{a_r-1}{r-2} + \cdots + \binom{a_s-1}{s-2}$$

Thus, from (2),

$$\begin{aligned} m([H]_{r-1}) &\geq m(H_1) + m([H_1]_{r-2}) \\ &\geq \binom{a_r-1}{r-1} + \cdots + \binom{a_s-1}{s-1} + \binom{a_r-1}{r-2} + \cdots + \binom{a_s-1}{s-2} \\ &= \binom{a_r}{r-1} + \cdots + \binom{a_s}{s-1} \end{aligned}$$

which is what we had to show.

Suppose now that

(4)
$$m(H_1) < \binom{a_r-1}{r-1} + \binom{a_{r-1}-1}{r-2} + \cdots + \binom{a_s-1}{s-1}$$

As a consequence we can write

$$\begin{aligned} m(H-H(x_1)) = m(H)-m(H_1) &> \binom{a_r}{r} + \cdots + \binom{a_s}{s} - \binom{a_r-1}{r-1} - \cdots - \binom{a_s-1}{s-1} \\ &= \binom{a_r-1}{r-1} + \binom{a_{r-1}-1}{r-1} + \cdots + \binom{a_s-1}{s} \end{aligned}$$

From (1), and applying the induction hypothesis on m to $H-H(x_1)$,

$$m(H_1) \geq m([H-H(x_1)]_{r-1}) \geq \binom{a_r-1}{r-1} + \binom{a_{r-1}-1}{r-2} + \cdots + \binom{a_s-1}{s-1}$$

which contradicts (4).

Corollary. *Let H be an r-uniform hypergraph and let k be an integer with $r > k \geq 2$. If a is the largest integer such that $m(H) \geq \binom{a}{r}$ then*

$$m([H]_k) \geq \binom{a}{k}$$

Proof. Let H_1 be a partial hypergraph of H with $m(H_1) = \binom{a}{r}$. From Theorem 14,

$$m([H_1]_{r-1}) \geq \binom{a}{r-1}$$

Let H_2 be a partial hypergraph of $[H_1]_{r-1}$ with $m(H_2) = \binom{a}{r-1}$. By Theorem 14,

$$m([H_2]_{r-2}) \geq \binom{a}{r-2},$$

etc. Finally, $m([H_{r-k}]_k) \geq \binom{a}{k}$. Since $[H]_k \supset [H_{r-k}]_k$ we also have

$$m([H]_k) \geq \binom{a}{k}.$$

Q.E.D.

7. Conformal Hypergraphs

We say that a hypergraph H is *conformal* if all the maximal cliques of the graph $[H]_2$ are edges of H. If H is simple, it is conformal if and only if the edges of H are the maximal cliques of a graph.

More generally, consider an integer $k \geq 2$. Every edge A of a hypergraph H satisfies the property: the edges of $[H]_k$ contained in A constitute a k-complete hypergraph. If every set $A \subset X$ maximal with this property is an edge of H the hypergraph is said to be *k-conformal.* Hence a hypergraph is conformal if and only if it is 2-conformal.

Proposition. *A hypergraph H is k-conformal if and only if for every set $A \subset X$ the following two conditions are equivalent:*

(C_k) *every $S \subset A$ with $|S| \leq k$ is contained in some edge of H,*

(C) *the set A is contained in an edge of H.*

Observe that (C) always implies (C_k).

Lemma. *A hypergraph is k-conformal if and only if its dual is k-Helly.*

Proof. In the hypergraph $H = (E_1, E_2, ..., E_m)$ the set $A = \{x_j / j \in J\}$ satisfies the condition (C_k) if and only if in the dual hypergraph $H^* = (X_1, X_2, ..., X_n)$ the set J satisfies

(D_k) $I \subset J,\ |I| \leq k$ implies $\bigcap_{i \in I} X_i \neq \emptyset$

Similarly, the set A satisfies Condition (C) if and only if in the dual hypergraph H^*, the set J satifies

$$(D) \qquad \bigcap_{j \in J} X_j \neq \emptyset.$$

Thus (C_k) is equivalent to (C) if and only if (D_k) is equivalent to (D).

Theorem 15. *A simple hypergraph H is k-conformal if and only if for each partial hypergraph $H' \subset H$ having $k+1$ edges, the set $\{x/x \in X, d_{H'}(x) \geq k\}$ is contained in an edge of H.*

Proof. From Theorem 10, the dual hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ is k-Helly if and only if for a set $F = \{e_j/j \in J\}$ with $|J| = k+1$, the intersection of the X_i with $|X_i \cap J| \geq k$ is non-empty. Or, again, for each $H' = \{E_j/j \in F\}$ with $|J| = k+1$ there exists an edge of H which contains the set

$$\{x/d_{H'}(x) \geq k\}.$$

Corollary (Gilmore's Theorem). *A necessary and sufficient condition for a hypergraph H to be conformal is that for any three edges E_1, E_2, E_3, the hypergraph H has an edge containing the set*

$$(E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$$

It suffices to put $k = 2$ in the statement of Theorem 15.

8. Representative Graphs

Given a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on X, its *representative graph*, or *line-graph* $L(H)$ is a graph whose vertices are points $e_1, e_2, \ldots, e_m$ representing the edges of H, the vertices e_i, e_j being adjacent if and only if $E_i \cap E_j \neq \emptyset$.

Example 1. The representative graph of a simple graph G was characterised by Beineke [1968]: a graph is an $L(G)$ if and only if it does not contain as an induced subgraph any of the graphs $G_1, G_2, \ldots, G_9$ shown in Figure 5.

Example 2. The representative graph of a multigraph G was characterised by Bermond and Meyer [1973]: a graph is an $L(G)$ if and only if it does not contain any of the graphs $G'_1, G'_2, \ldots, G'_7$ shown in Figure 6.

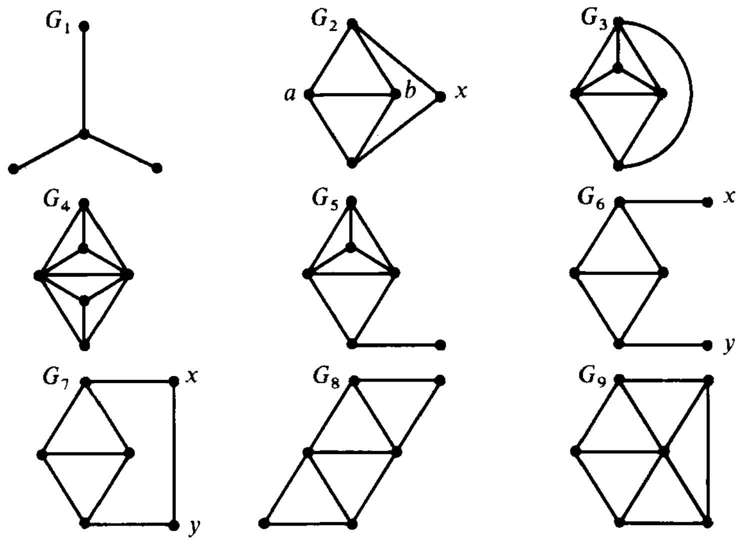


Figure 5. *The 9 forbidden configurations for the representative graph of a simple graph.*

Example 3. The representative graph of a multigraph without triangles or loops is characterised by: each vertex appears in at most 2 maximal cliques.

Example 4. The representative graph of a bipartite multigraph is characterised by: each vertex appears in at most two maximal cliques, and every elementary odd cycle contains two sides of a triangle.

Example 5. If H is a family of intervals on a line, there is a characterisation of $L(H)$ due to Gilmore and Hoffman (cf. *Graphs*, Chapter 16, Theorem 12) : it is a triangulated complement of a comparability graph. This concept has a simple interpretation: if m individuals were present during various intervals of time in a meeting room, a detective who demands of each person whom he has met can trace the "graph of meetings": if nobody lies, the graph represents a family of intervals.

We do not know any similar characterisation for the representative graph of a family of convex sets in the plane, but we do know that every graph represents convex sets in the 3-dimensional space (Wegner [1965]).

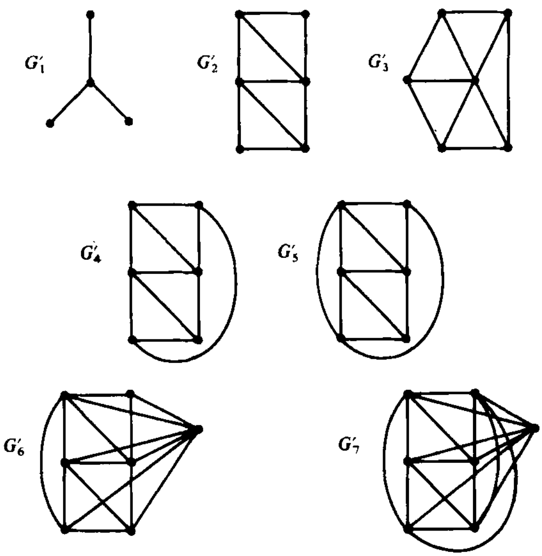


Figure 6. *The 7 forbidden configurations for the representative graph of a multigraph.*

Proposition 1. *The representative graph of a hypergraph H is the 2-section $[H^*]_2$. Further, the following two properties are equivalent:*

(i) *H satisfies the Helly property and G is the representative graph of H;*

(ii) *the maximal edges of H^* are the maximal cliques of G.*

Clearly the graph $[H^*]_2$ is isomorphic to $L(H)$, but $[H^*]_2$ can have loops if H^* has loops.

For the other part, if H has the Helly property, H^* is conformal; thus (i) implies that $G = [H^*]_2$ has as cliques the maximal edges of H^*. Similarly (ii) implies (i).

Observe that if $G = L(H)$ and if H does not satisfy the Helly property it can happen that H^* does not contain the maximal cliques of $L(H)$. For example, if H is the hypergraph H_2 in Figure 8, $L(H)$ is the graph G in Figure 8; the maximal cliques of G are not the edges of H^*.

Proposition 2. *Every graph is the representative graph of a linear hypergraph.*

A simple graph G on $\{x_1,...,x_n\}$ is the representative graph of a linear hypergraph $(X_1,X_2,...,X_n)$ if we take for X_i the set of edges of G adjacent to the vertex x_i.

Proposition 3. *A graph G is the representative graph of an r-uniform hypergraph if and only if G contains a family $\mathcal{C}$ of cliques with the following properties:*

(Π_0) *each clique of $\mathcal{C}$ is of cardinality ≥ 2;*

(Π_1) *every edge of G is contained in at least one clique of $\mathcal{C}$;*

(Π_2) *each vertex of G appears in at most r cliques of $\mathcal{C}$;*

(Π_3) *for each vertex x which is covered by exactly cliques of $\mathcal{C}$, the intersection of these cliques is $\{x\}$.*

Indeed, consider the r-regular hypergraph $\mathcal{C}'$ obtained from $\mathcal{C}$ by adjoining loops, which is always possible because of (Π_2). Let H be the dual of the hypergraph $\mathcal{C}'$. By (Π_1) we have $L(H) = [H^*]_2 = [\mathcal{C}] = G$. By ($\Pi_3$) the hypergraph H has no repeated edges: it is thus an r-uniform hypergraph.

Proposition 4. *A graph G is the representative graph of a linear r-uniform hypergraph if and only if, in G there exists a family $\mathcal{C}$ of cliques satisfying* (Π_0), Π_2) *and*

(Π_1') *each edge is contained in exactly one clique of $\mathcal{C}$.*

Let $\mathcal{C}'$ be the r-regular hypergraph obtained from $\mathcal{C}$ by adding loops, which is possible from (Π_2). Let H be the dual hypergraph of $\mathcal{C}'$. From (Π_1'), $L(H) = G$, the hypergraph $\mathcal{C}$ is linear and hence its dual H is linear (Proposition 3, §3).

One can ask if it is possible to characterise $L(H)$ by a finite family of forbidden subgraphs in the case $r \neq 2$. In fact, Nickel, then Gardner, then Bermond, Germa, Sotteau [1977] exhibited an infinite family of forbidden configurations for a representative graph of a 3-uniform hypergraph.

The graphs $G_1(t)$, $G_2(t)$, $G_3(t)$ of Figure 7 constitute infinite families of minimal excluded configurations for the representative graph of a 3-uniform linear hypergraph.

Nonetheless, it can be shown that

Theorem 16 (Naik, Rao, Shrikhande, Singhi [1982]). *There exists a finite family $\mathcal{F}_3$ of graphs such that every graph G with minimum degree ≥ 69 is the representative*

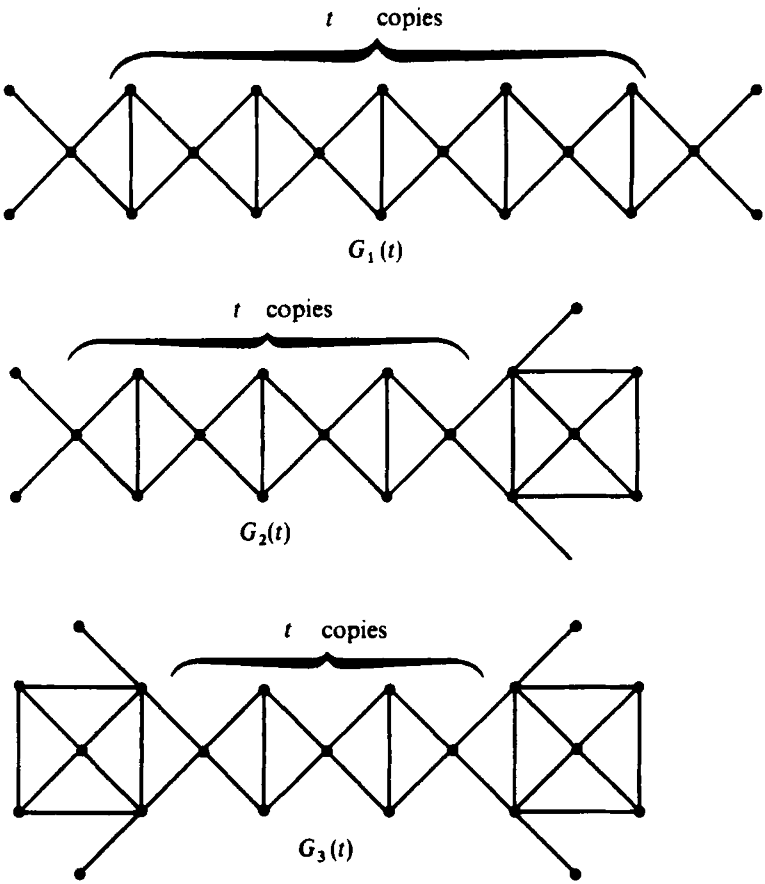


Figure 7

graph of a linear 3-uniform hypergraph if and only if G contains no member of $\mathcal{F}_3$ as an induced subgraph.

More generally, they show the existence of a cubic polynomial $f(k)$ with the property that for each k there exists a finite family $\mathcal{F}_k$ of forbidden graphs such that every graph G of minimum degree $\geq f(k)$ is the representative graph of a linear k-uniform hypergraph if and only if G does not contain a member of $\mathcal{F}_k$ as an induced subgraph.

By way of example, we can check the preceding propositions on the graph G of Figure 8 which is, at one and the same time, the representative graph of the hypergraphs H_1, H_2 and H_3 of Figure 8.

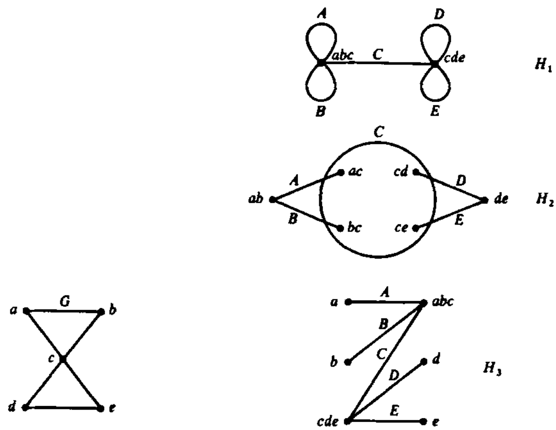


Figure 8

We shall denote by $\Omega(G)$ the minimum order of those hypergraphs H with $G = L(H)$; for example, for the graph G in Figure 8, $\Omega(G) = 2$ since $G = L(H_1)$.

The determination of $\Omega(G)$ brings us back to the determination of the chromatic number by the following result.

Lemma. *Let G be a graph on $\{x_1,x_2,...,x_n\}$ without isolated vertices, and let $\overline{G}$ be the graph whose vertices correspond to the edges of G, the vertices corresponding to the edges $[a,b]$ and $[x,y]$ of G being adjacent if and only if $\{a,b,x,y\}$ is not a clique in G (i.e. at least one of ax,ay,bx,by is not an edge of G). Then the minimum order $\Omega(G)$ of the hypergraphs for which G is the representative graph is equal to the chromatic* *number of* $\overline{G}$.

Proof.

1. We shall show that to each q-colouring $(\overline{S}_1, \ldots, \overline{S}_q)$ of the vertices of $\overline{G}$ with q colours we may associate a hypergraph $H = (X_1, X_2, \ldots, X_n)$ of order q such that $G = L(H)$.

Indeed, the set $\overline{S}_i$ of vertices of $\overline{G}$ coloured with colour i is stable; if $[a,b]$ is an edge of G belonging to $\overline{S}_i$, the vertex a is adjacent to each end of any edge in $\overline{S}_i$. The ends of the edges of $\overline{S}_i$ thus generate a clique E_i of G. The hypergraph $\mathcal{C} = (E_1, E_2, \ldots, E_q)$ is such that each edge and each vertex of G is covered by at least one of the E_i. Thus the dual hypergraph $H = (X_1, X_2, \ldots, X_n)$ of $\mathcal{C}$ satisfies $L(H) = [\mathcal{C}]_2 = G$, and H is of order q.

2. We shall show that to each hypergraph $H = (X_1, \ldots, X_n)$ of order q for which $G = L(H)$, we may associate a q-colouring $(S_1, S_2, \ldots, S_q)$ of the vertices of $\overline{G}$. Indeed, denote by E^k the set of vertices of H which belong to exactly k of the sets X_i. We have

$$q = |E^1| + |E^2| + |E^3| + \cdots$$

To each $e \in E^1$, which belongs to exactly one set $X_{i(e)}$, associate the 1-clique $\{x_{i(e)}\}$; to each $e \in E^2$, which belongs to exactly two sets $X_{i(e)}$ and $X_{j(e)}$, associate the 2-clique $\{x_{i(e)}, x_{j(e)}\}$ of G; to each e of E^3 belonging to exactly three sets $X_{i(e)}$, $X_{j(e)}$, $X_{k(e)}$ associate the 3-clique $\{x_{i(e)}, x_{j(e)}, x_{k(e)}\}$ of the graph G; etc.

We have thus defined in G a family $(E_1, E_2, \ldots, E_q)$ of q cliques and it is evident that each edge $[x_i, x_j]$ of G belong to at least one of these (since $X_i \cap X_j$ contains a point of H). Denote by $\overline{S}_1$ the set of edges of G contained in the clique E_1, by $\overline{S}_2$ the set of edges of G contained in E_2 which are not already contained in E_1; etc. The family $(\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_q)$ is then a q-colouring the vertices of $\overline{G}$.

It follows from points 1 and 2 that the chromatic number of $\overline{G}$ is equal to the least order of a hypergraph H such that $G = L(H)$.

Theorem 17. *Let* G *be a simple graph without isolated vertices, without triangles, with* m *edges; the minimum order of the hypergraphs for which* G *is the representative graph is* $\Omega(G) = m$.

Indeed, the graph $\overline{G}$ defined in the lemma is the clique K_m; the minimum order $\Omega(G)$ is thus m, the chromatic number of $\overline{G}$.

Theorem 18 (Erdös, Goodman, Pósa [1966]). *Let G be a graph of order n without isolated points; then*

(1) $$\Omega(G) \leq [n^2/4].$$

Further, for each n, this bound is the best possible.

Proof.

1. Indeed, we know (cf. *Graphs*, Theorem 5, Chapter 1) that we can always cover the edges and the vertices of a graph G by a family of 2-cliques and 3-cliques $\boldsymbol{C} = (E_1, E_2, \ldots, E_k)$ with $k \leq [\frac{n^2}{4}]$; since $G = [\boldsymbol{C}]_2$ is the representative graph of the dual of the hypergraph $\boldsymbol{C}$, and since this dual is of order k, we have

$$\Omega(G) \leq k \leq [\frac{n^2}{4}],$$

which gives us (1).

2. We show that for every n, we can have equality in (1).

If $n = 2k$ is even, take for G the complete bipartite graph $K_{k,k}$; since it has no triangles or isolated vertices we have, from Theorem 17

$$\Omega(K_{k,k}) = k^2 = \frac{n^2}{4} = [\frac{n^2}{4}].$$

If $n = 2k+1$ is odd, take for the G bipartite complete graph $K_{k,k+1}$ which gives

$$\Omega(K_{k,k+1}) = k(k+1) = \frac{(n-1)}{2}\frac{(n+1)}{2} = \frac{n^2-1}{4}$$
$$= [\frac{n^2}{4}].$$

Thus we can have equality in (1).

Exercises on Chapter 1

Exercise 1 (§1)

Give conditions that a simple graph must satisfy in order that is dual is also a simple graph.

Exercise 2 (§1)

Define an "interval hypergraph" to be a hypergraph whose vertices can be represented by points on a line in such a way that the edges are intervals of the line. Show that if an interval hypergraph is simple then its dual is also an interval hypergraph. Show that a subhypergraph of an interval hypergraph is an interval hypergraph.

Exercise 3 (§1)

For two integers $n \geq r \geq 2$ the r-uniform complete hypergraph of order n is the hypergraph K_n^r whose vertex set is a set X of cardinality n, and whose edges are all the r-subsets of X. What is the rank of K_n^r and of its dual $(K_n^r)^*$?

Exercise 4 (§3)

Let H be an intersecting family of order n, of rank $r = \max_i |E_i|$ and anti-rank $s = \min_i |E_i|$. Hilton [1975] showed that

$$m(H) \leq \sum_{i=s}^{r} \binom{n-1}{r-1}.$$

Show that this result generalises the Erdős, Chao-Ko, Rado Theorem.

Exercise 5 (§3)

Show how Theorem 6 implies relation (1) of Sperner's Theorem (Theorem 2).

Exercise 6 (§3)

Let $H = \{E_1, E_2, \ldots, E_m\}$ be a hypergraph satisfying

$$E_j \not\subset E_k \qquad (j \neq k)$$

$$E_j \cap E_k \neq \emptyset$$

$$E_j \cup E_k \neq X.$$

Show that $H' = (E_1,E_2,...,E_m,X-E_1,...,X-E_m)$ is a simple hypergraph. Deduce the following inequality (Schönheim [1968]):

$$m(H) \leq \frac{1}{2}\binom{n}{[n/2]},$$

and this bound is best possible.

Exercise 7 (§3)

Show as in the lemma:

Let $\mathcal{A} = (A_1,A_2,...,A_m)$ be a family of m circular intervals on a circle of n points with

(i) $|A_i| > n/2$;

(ii) $A_i \cap A_j \neq \emptyset \quad (i \neq j)$

(iii) $A_i \not\subset A_j \quad (i \neq j)$

Then we have $m \leq n$, with equality if $\mathcal{A}$ is the family $\mathcal{A}_k$ of distinct circular intervals having fixed cardinality $k > \frac{n}{2}$.

Exercise 8 (§3)

Let $\mathcal{A}$ be a family of circular intervals satisfying conditions (2) and (3) of Exercise 7, and for $A \in \mathcal{A}$, put:

$$p(A) = \frac{n-|A|+1}{|A|} \text{ if } |A| \leq \frac{n}{2}$$

$$= 1 \text{ if } |A| > \frac{n}{2}.$$

Show that $\Sigma p(A) \leq n$.

Exercise 9 (§3) (Open Problem)

Let H be a hypergraph on X of order n, let $k \geq 2$ and $t \leq n$ be integers. Erdös and Frankl [1979] conjectured that

$$I \subset \{1,2,...,m\}, \quad |I| = k$$

implies

$$|\bigcup_{i \in I} E_i| \leq n-t$$

and if m is the maximum with this condition then $H = \{F/F \subset X, |F \cap Y| \leq s\}$ for an integer s and for a set Y of cardinality $t+ks$.

Katona showed that the conjecture is true if $k = 2$, $t \neq 1$. Frankl [1979] showed that the conjecture is true for $k > 2$, $t < \frac{k2^k}{150}$.

Exercise 10 (§4)

Show, using the methods of proof of Theorem 7, that if H is an hereditary hypergraph, the graph $\overline{L(H)}$ (complement of the representative graph) admits a matching covering every vertex, except at most one in each connected component of odd order (Berge [1976]).

Exercise 11 (§5)

Show, using Theorem 5, that the dual of an interval hypergraph has the Helly property.

Exercise 12 (§5)

Consider integers $a_1 < m_1, a_2 < m_2, \ldots, a_k < m_k$. Show that the system

$$x \equiv a_i \ mod \ m_i \quad \text{for} \quad i = 1,2,\ldots,k$$

has a solution x if and only if every pair (i,j) with $1 \leq i < j \leq k$ satisfies

$$a_i \equiv a_j \ mod \ lcm \ (m_i, m_j).$$

(Use Corollary to Theorem 10).

Exercise 13 (§5)

Show that for $k \geq 3$, every simple graph is k-Helly.

Exercise 14 (§6)

Using Frankl's lemma (Lemma 2 of Theorem 14), prove the following result, due to Lovász (which generalises the corollary to Theorem 14):

Let H be an r-uniform hypergraph and let x be a positive real number such that

$$m(H) = \frac{x(x-1)...(x-r+1)}{r!}$$

Then

$$m([H]_{r-1}) \geq \frac{x(x-1)...(x-r+2)}{(r-1)1}.$$

Exercise 15 (§8)

Let $d(m)$ be the minimum cardinality of a set X having the property that every graph of order m is the representative graph of at least m distinct subsets of X. Show (by induction on m) that

$$d(2) = 2$$

$$d(3) = 3$$

$$d(m) = [\frac{m^2}{4}] \text{ if } m \geq 4$$

(Erdös, Goodman, Pósa [1966]).

Chapter 2

Transversal Sets and Matchings

1. Transversal hypergraphs

Let $H = (E_1,...,E_m)$ be a hypergraph on a set X. A set $T \subset X$ is a *transversal* of H if it meets all the edges, that is to say:

$$T \cap E_i \neq \emptyset \qquad (i = 1,2,...,m)$$

The family of minimal transversals of H constitutes a simple hypergraph on X called the *transversal hypergraph* of H, and denoted by $Tr\,H$.

Example 1. If the hypergraph is a simple graph G, a set S is *stable* if it contains no edge, that is, if its complement $X-S$ meets all the edges of G. Thus,

$$Tr\,G = \{X-S/S \text{ is a maximal stable set of } G\}.$$

Example 2. The complete r-uniform hypergraph K_n^r on X admits as minimal transversals all the subsets of X with $n-r+1$ elements. Thus

$$Tr(K_n^r) = K_n^{n-r+1}$$

Example 3. Let us consider the complete r-partite hypergraph $K^r_{n_1,n_2,...,n_r}$ in which the set of vertices is $X^1 \cup X^2 \cup \cdots \cup X^r$ and the edges are the r-tuples $\{x^1,x^2, \ldots ,x^r\}$ with $x^1 \in X^1, x^2 \in X^2,...,x^r \in X^r$. Clearly $X^1,X^2,...,X^r$ are all minimal transversals. If there existed a minimal transversal $T \neq X^1,X^2,...,X^r$, there would exist for every i a vertex $a_i \in X^i - T$. The set $\{a_1,a_2,...,a_r\}$ would not meet T, and since it is an edge of the hypergraph, we have a contradiction. Therefore there are no other minimal transversals besides $X^1,X^2,...,X^r$, and consequently:

$$Tr(K^r_{n_1,n_2,...,n_r}) = (X^1,X^2,...,X^r).$$

Example 4. Let G be a transport network, i.e. a directed graph with a "source" a and a "sink" z (cf. *Graphs*, Chap. 6). An edge of H would be a set of arcs of G making up an elementary path from a to z. Clearly, H is a simple hypergraph, and $Tr\,H$

is the set of minimal "cuts" between a and z.

Generalizing the "arc-colouring lemma" which has proved its effectiveness in the study of transport networks (example 4), we can state:

Vertex-colouring lemma. *Let* $H = (E_1, E_2, \ldots)$ *and* $H' = (F_1, F_2, \ldots)$ *be two simple hypergraphs on a set* X. *Then* $H' = Tr\,H$ *if and only if every pair* (A, B) *with* $A, B \subset X$, $A \cup B = X$, $A \cap B = \emptyset$, *satifies:*

(i) *there exists either an* $E \in H$ *contained in* A *or an* $F \in H'$ *contained in* B;

(ii) *these two cases cannot happen simultaneously.*

Proof.

1. Let $H' = Tr\,H$, and consider a bipartition (A, B) of X. If A contains an $E \in H$, we have (i). If not, then $X - A = B$ is a transversal of H and therefore contains a minimum transversal $T \in Tr\,H$. Thus T is an edge F of H' and $F \supset B$; we therefore again have (i). Moreover (ii) is obvious.

2. Let H' and H'' be two simple hypergraphs such that every pair (A, B) satisfies (i) and (ii) with H and H' on the one hand, and H and H'' on the other. We show that this implies $H' = H''$. (As we have (i) and (ii) with H and $H'' = Tr\,H$ from (1), this certainly shows that $H' = Tr\,H$).

If not, there exists a set $F' \in H' - H''$. As the pair $(X - F', F')$ satisfies (ii) with H, H', there is no edge $E \in H$ contained in $X - F'$; and as the pair $(X - F', F')$ satisfies (i) with H, H'', there exists an $F'' \in H''$ such that $F'' \subset F'$. On the other hand $X - F''$ does not contain an edge $E \in H$, (as above); since the pair $(X - F'', F'')$ satisfies (i) with H and H', there exists a $F_1' \in H'$ with $F_1' \subset F''$.

Thus, $F_1' \subset F'' \subset F'$; and as H' is a simple hypergraph $F_1' = F'$, thus $F' \in H''$: a contradiction. By symmetry there cannot exist a set $F'' \in H'' - H'$ either.

Therefore $H' = H''$.

If we take for H'' the hypergraph $Tr\,H$, which is possible from (1), we get $H' = Tr\,H$, which gives the proof.

Corollary 1. *Let* H *and* H' *be two simple hypergraphs. Then* $H' = Tr\,H$ *if and only if* $H = Tr\,H'$.

Indeed $H' = Tr\,H$ if and only if every pair (A,B) satisfies (i) and (ii) with H,H'; that is every pair (B,A) satisfies (i) and (ii) with H',H; that is $H = Tr\,H'$.

Corollary 2. *Let H be a simple hypergraph. Then $Tr(Tr\,H) = H$.*

(From Corollary 1).

Application: Problem of the keys of the safe. An administrative council is composed of a set X of individuals. Each of them carries a certain weight in decisions, and it is required that every set $E \subset X$ carrying a total weight greater than some threshold fixed in advance, should have access to documents kept in a safe with multiple locks. The minimal "coalitions" which can open the safe constitute a simple hypergraph H. The problem consists in determining the number of locks necessary so that by giving one or more keys to every individual, the safe can be opened if and only if at least one of the coalitions of H is present.

If $Tr\,H = (F_1,F_2,\ldots,F_m)$, and if the key to the i-th lock is given to all the members of F_i, it is clear that every coalition $E \in H$ would be able to open the safe; on the other hand, if $A \subset X$ does not contain any edge of H, the individuals making up the set A will not be able to open the safe, since A is not a transversal of $Tr\,H$ (Corollary 2). The minimum number of locks that are necessary is therefore $m(Tr\,H)$. In particular if all the n members of the administrative council have the same weight, and if the presence of r individuals is necessary in order to open the safe, the number of locks necessary is

$$m(K_n^{n-r+1}) = \binom{n}{n-r+1}.$$

We now propose to study the transversal hypergraph of an intersecting hypergraph. If H and H' are two simple hypergraphs on X, we write $H \subset H'$ if every edge of H is also an edge of H'; we write $H = H'$ if $H \subset H'$ and $H' \subset H$. We write $H \prec H'$ if every edge of H contains an edge of H'. Therefore:

$$H \subset H' \Rightarrow H \prec H'.$$

Finally we denote by a $\chi(H)$ the *chromatic number* of H, that is to say the smallest number of colours necessary to "colour" the vertices of H such that no edge of cardinality > 1 is monochromatic.

Lemma 1. *If H and H' are simple hypergraphs on X, then*

$$\left.\begin{array}{l} H \prec H' \\ H' \prec H \end{array}\right\} \Rightarrow H = H'.$$

Indeed, since $H \prec H'$, every edge E_i of H contains an edge F of H'; since $H' \prec H$, the edge F of H' contains an edge E_j of H. Hence

$$E_i \supset F \supset E_j.$$

Since H is a simple hypergraph, $i = j$, and hence every edge of H is an edge of H'. By symmetry, $H = H'$.

Lemma 2. *A simple hypergraph H without loops satisfies $\chi(H) > 2$ if and only if $Tr\,H \prec H$.*

Indeed, if $\chi(H) > 2$, we have $Tr\,H \prec H$. Otherwise there exists a $T \in Tr\,H$ containing no edge of H. But then the bipartition $(T, X-T)$ is such that no edge of H is contained in a single class; it is therefore a bicolouring of H, and that contradicts $\chi(H) > 2$.

Conversely, if $Tr\,H \prec H$, we have $\chi(H) > 2$. Otherwise there exists a bicolouring (A,B) of the vertices of H. From the vertex colouring lemma, B contains a set $T \in Tr\,H$, and since $Tr\,H \prec H$, we have also $B \supset E$ for an $E \in H$, which contradicts the fact that (A,B) is a bicolouring of H.

Lemma 3. *A hypergraph H is intersecting if and only if $H \prec Tr\,H$.*

For if H is intersecting, every $E \in H$ is a transversal of H, and therefore E contains a minimal transversal $T \in Tr\,H$, so $H \prec Tr\,H$.

Conversely, if $H \prec Tr\,H$, every $E \in H$ contains a transversal of H, and therefore meets all the edges of H, that is, H is intersecting.

Theorem 1. *A simple hypergraph H without loops satisfies $H = Tr\,H$ if and only if:*

(i) $\chi(H) > 2$;

(ii) *H is intersecting.*

This is obvious from Lemmas 1, 2 and 3.

Corollary. *Let H be a simple intersecting hypergraph without loops. Then either $\chi(H) = 2$, or $\chi(H) = 3$ and every hypergraph H' obtained from H by replacing an edge E by a new edge of the form $E \cup \{x\}$ with $x \in X - E$ is bicolourable.*

For if $\chi(H) > 2$, we have $H = Tr\,H$ from Theorem 1. As E is a transversal set of H, and hence of H', we have $E \cup \{x\} \notin Tr\,H'$ so that $H' \neq Tr\,H'$ and hence $\chi(H') = 2$, from Theorem 1.

A 3-colouration of H can be obtained from a bicolouring of H' by replacing the colour of a $y \in E$ by a third colour not already used. Therefore $\chi(H) = 3$.

We give a few examples of hypergraphs H for which $H = Tr\,H$.

Example 1. The complete r-uniform hypergraph K^r_{2r-1} satisfies $Tr(K^r_{2r-1}) = K^r_{2r-1}$.

Example 2. The finite projective plane P_7 on 7 points satisfies $Tr(P_7) = P_7$, for it is an intersecting family and non-bicolourable: If one wanted to colour the vertices with two colours + and -, the last vertex to be coloured could not be given either + or - (cf. Figure 1).

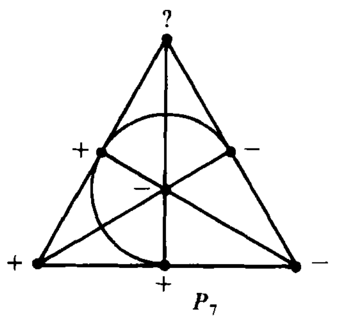


Figure 1.

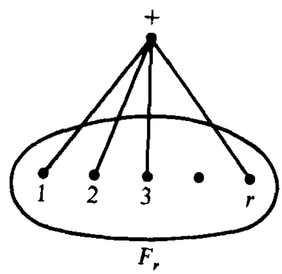


Figure 2.

Example 3. The "fan" of rank r is a hypergraph F_r having r edges of cardinality 2 and one edge of cardinality r, arranged as in Figure 2. It is an intersecting family and non-bicolourable; therefore $Tr(F_r) = F_r$.

Example 4. Lovasz's hypergraph L_r is a hypergraph defined by r sets of vertices $X^1 = \{x^1_1\}$, $X^2 = \{x^2_1, x^2_2\}$, $X^3 = \{x^3_1, x^3_2, x^3_3\}$, $\cdots$, $X^r = \{x^r_1, x^r_2, \ldots, x^r_r\}$, and having as edges all the sets of the form

$$X^i \cup \{x^{i+1}_{k_1}, x^{i+2}_{k_2}, \ldots, x^r_{k_r}\},$$

Clearly, L_r is an intersecting family. Moreover $\chi(L_r) > 2$. Otherwise there exists a bicolouring (A,B), and at least one of the sets X^i is monochromatic (in particular X^1, which has cardinality 1); let i be the largest integer such that X^i is monochromatic. Then there exists a monochromatic edge of the form $X^i \cup \{x_{k_1}^{i+1},...,x_{k_r}^r\}$, which contradicts the fact that (A,B) is a bicolouring of L_r.

Therefore, by virtue of Theorem 1, $Tr(L_r) = L_r$.

Example 5. In the same way, using Theorem 1, we show that the hypergraph $\overline{L}_3 = (X-E/E \in L_3)$ satisfies $Tr\,\overline{L_3} = \overline{L_3}$.

Example 6. The "generalised fan" is a hypergraph H having as edges r distinct sets $E_1, E_2, ..., E_r$ with $E_i \cap E_j = \{x_0\}$ for $i \neq j$ and $2 = |E_1| \leq |E_2| \leq \cdots \leq |E_r|$, to which are added the edges of the complete r-partite hypergraph on $(E_1 - \{x_0\}, E_2 - \{x_0\}, \cdots, E_r - \{x_0\})$. We show in the same way that $Tr\,H = H$.

We shall represent by a diagram the different envisaged properties which generalise, for a hypergraph H, the relation $H = Tr\,H$. We shall prove those implications in this diagram which have not already been proved by the preceding propositions.

Proposition 1. *For a simple hypergraph H, the following two conditions are equivalent:*

(i) *H has no loops and $\chi(H) > 2$;*

(ii) *$Tr\,H$ is intersecting and is not a star.*

For if (i) holds, then $Tr\,H \prec H$ (from Lemma 2), and the hypergraph $H' = Tr\,H$ is not a star. Thus $H' = Tr\,H \prec H = Tr\,H'$ and hence H' is intersecting (from Lemma 3). The converse is proved in the same way.

Proposition 2. *Every hypergraph H with property (7) satisfies property (8).*

We note that if H satisfies property (7) it has no loops and is simple.

Since $\chi(H-E) = 2$, there exists a bicolouring (A,B) of $H-E$, and E is monochromatic in this bicolouring. Suppose for example that $E \subset A$. If we change the colour of an arbitrary point x of E, a new edge $E' \in H$ will become coloured B, whence $E \cap E' = \{x\}$. From this (8) follows.

Proposition 3. *Every simple hypergraph H without loops having property (2) satisfies property (8).*

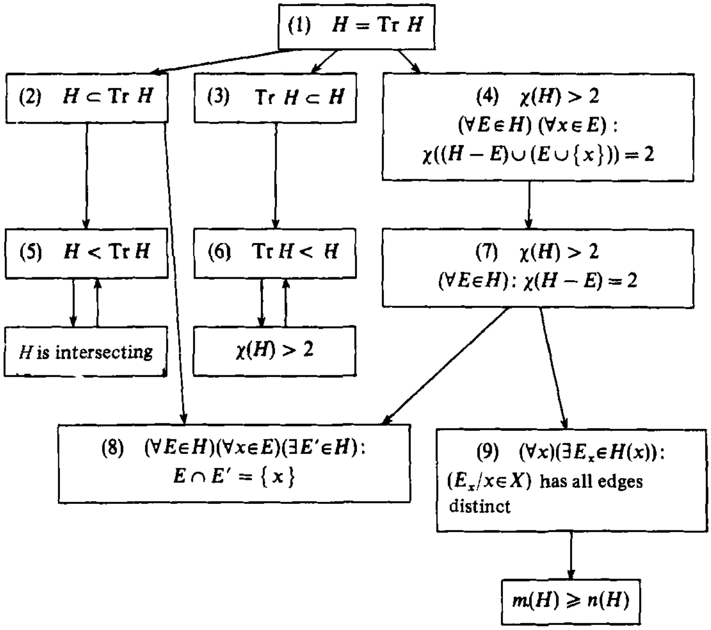


Figure 3.
(H simple and without loops)

Since every $E \in H$ is a minimal transversal of H, the set $E-\{x\}$ is disjoint with some edge $E' \in H$, whence $E \cap E' = \{x\}$. From this (8) follows.

Proposition 4 (Seymour [1974]). *Let H be a hypergraph on X with property (7) and let $A \subset X$; then there is no bipartition (A_1,A_2) of A into two transversal sets of H_A.*

We note that since H satisfies property (7), it has no loops and is simple. Suppose that such a bipartition (A_1,A_2) exists and consider the partial hypergraph $H' = (E/E \in H, E \cap A = \emptyset)$. We have $H' \neq \emptyset$, for if not then (A_1,A_2) would extend to a bicolouring of H. We have $H' \neq H$, since $A \neq \emptyset$. Thus from property (7), the hypergraph H' has a bicolouring (B_1,B_2) and $B_1 \cup B_2 \subset X-A$. Since H has no loops, $E \in H'$ implies

$$E \cap B_1 \neq \emptyset, \quad E \cap B_2 \neq \emptyset.$$

Furthermore $E \in H - H'$ implies

$$E \cap A_1 \neq \emptyset, \quad E \cap A_2 \neq \emptyset.$$

Thus $(A_1 \cup B_1, A_2 \cup B_2)$ generates a bicolouring of H, which contradicts (7).

Proposition 5 (Seymour [1974]). *Let H be a hypergraph on X with property (7). Every $A \subset X$ meets at least $|A|$ edges of H, with equality possible only if $A = \emptyset$ or $A = X$.*

(*) Proof. We consider three cases.

Case 1. $A = \emptyset$; the result is trivial.

Case 2. $A = X$; the incidence matrix M of H defines a system of $m(H) = m$ linear equations: $M^*\mathbf{z} = \mathbf{0}$. If $m < |X| = n$, we have m linear equations with $n > m$ unknowns, and hence there exists a solution $(z_1, z_2, \ldots, z_n) \neq \mathbf{0}$.

Let $A = \{x_i / z_i \neq 0\}$, $A^+ = \{x_i / z_i > 0\}$, $A^- = \{x_i / z_i < 0\}$.

Clearly (A^+, A^-) is a bipartition of A into two transversal sets of H_A, which contradicts Proposition 4. Hence $m \geq n$, and the result follows.

Case 3. $A \neq X$, $A \neq \emptyset$. We put

$$H' = \{E / E \in H, E \subset A\}$$

$$H'' = \{E / E \in H, E \cap A = \emptyset\}.$$

Since $A \neq X$, $A \neq \emptyset$, we have $H' \neq H$, $H'' \neq H$. Thus there exists, from (7), a bicolouring (A_1, A_2) of H' and a bicolouring (B_1, B_2) of H''. Since $(A_1 \cup B_1, A_2 \cup B_2)$ cannot define a bicolouring of H (since $\chi(H) > 2$) we have

$$H \neq H' \cup H''$$

Thus there is an edge $E_0 \in H - (H' \cup H'')$ that is to say with:

$$(1) \qquad \begin{cases} E_0 \not\subset A \\ E_0 \cap A \neq \emptyset. \end{cases}$$

Suppose that the set A does not meet more than $|A|$ edges of H. We see as in Case 2 that there exists on A a real function $z(x)$, not identically zero, such that

$$\sum_{x \in E \cap A} z(x) = 0 \qquad (E \in H - E_0)$$

Put $\bar{z}(x) = z(x)$ if $x \in A$ and $\bar{z}(x) = 0$ if $x \notin A$. Then

$$\sum_{x \in E} \bar{z}(x) = 0 \qquad (E \in H - E_0).$$

We cannot have $\sum_{x \in E_0} \bar{z}(x) = 0$, since the sets $A^+ = \{x/\bar{z}(x)>0\}$ and $A^- = \{x/\bar{z}(x)<0\}$ would contradict Proposition 4. Suppose for example that

$$\sum_{x \in E_0} \bar{z}(x) > 0.$$

We then have, by virtue of Proposition 4, $E_0 \cap A^- = \emptyset$.

The hypergraph $H_1 = \{E/E \in H, E \subset X - (A^+ \cup A^-)\}$ is bicolourable (since $H_1 \neq H$), and admits a bicolouring (B_1, B_2).

The set $A^+ \cup B_1$ is a transversal of H; for we have either $E \in H_1$ or $E \cap A^+ \neq \emptyset$. Since $Tr\, H \subset H$, there exists an edge $E_1 \in H$ contained in $A^+ \cup B_1$.

If $E_1 \subset B_1$, then $E_1 \in H_1$, which contradicts the fact that (B_1, B_2) is a bicolouring of H_1. Hence $E_1 \cap A^+ \neq \emptyset$, and consequently

$$\sum_{x \in E_1} \bar{z}(x) > 0.$$

Thus $E_1 = E_0$, and consequently

$$E_0 \subset A^+ \cup B_1. \tag{2}$$

By the same arguments we obtain

$$E_0 \subset A^+ \cup B_2. \tag{3}$$

As B_1 and B_2 are disjoint, (2) and (3) give $E_0 \subset A^+ \subset A$, which contradicts (1).

Proposition 6. *Every hypergraph H with property (7) satisfies property (9).*

For the preceding proposition shows that the bipartite graph $G = (X, H; \Gamma)$ of the vertex-edge incidence of a hypergraph H with property (7) satisfies $|\Gamma A| \geq |A|$ for every $A \subset X$. From König's Theorem, this condition implies that to every $x \in X$ we can make correspond an edge $E_x \in H(x)$ such that the E_x are distinct edges. Then (7) implies the condition (9).

We deduce that $m(H) \geq n(H)$. The case where $m(H) = n(H)$ is characterised by the following theorem.

Theorem 2 (Seymour [1974]). *Let H be a hypergraph with property (7), and with $m(H) = n(H)$. Consider for every $x \in X$ an edge $E_x \in H(x)$ such that the E_x for $x \in X$ are distinct edges. Then the directed graph G defined on X by making an arc from x to y if $y \in E_x$, is strongly connected and has no even elementary circuits. Conversely, if $G = (X,\Gamma)$ is a directed graph on X which is strongly connected and without even elementary circuits, the hypergraph $H_G = (\{x\} \cup \Gamma x / x \in X)$ is a hypergraph on X with property (7) and with $m(H_G) = n(H_G)$.*

The proof arises from the previous propositions (cf. Seymour [1974]).

Corollary. *If H satisfies property (7) with $m(H) = n(H)$, then its dual H^* also satisfies property (7) with $m(H^*) = n(H^*)$.*

For in this case the maximum matching of the bipartite vertex-edge incidence graph establishes a bijection between the set of vertices of H and the set of edges of H. The graphs G_H and G_{H^*} therefore have the same properties.

Algorithm to determine TrH. If $H = (E_1, E_2, \ldots, E_m)$ and $H' = (F_1, F_2, \ldots, F_{m'})$ are two hypergraphs, put:

$$H \cup H' = (E_1, E_2, \ldots, E_m, F_1, F_2, \ldots, F_{m'})$$

$$H \vee H' = (E_i \cup F_j / i \leq m,\ j \leq m')$$

$$Min\ H = (E / E \in H; (\forall F \in H, F \subset E) \colon F = E)$$

Hence we have

$$(1) \qquad Tr(H \cup H') = Min(Tr\,H \vee Tr\,H')$$

Indeed, T_0 is a transversal of $H \cup H'$ if and only if T_0 is a transversal of H and a transversal of H', i.e.

$$T_0 \supset T \cup T',\ T \in Tr\,H,\ T' \in Tr\,H'.$$

Or, equivalently:

$$T_0 \in Tr\,H \vee Tr\,H'.$$

The formula (1) follows.

No polynomial algorithm for determining $Tr\,H$ is known (it belongs to the class of NP-complete problems). Nevertheless, for hypergraphs with a few vertices we have at hand many methods that are sufficiently effective (Maghout [1966], Lawler [1966], Roy [1970], etc.). We could use formula (1) in the following manner:

Put $H = (E_1, E_2, \dots, E_m)$ and $H_i = (E_1, E_2, \dots, E_i)$. Determine successively $Tr\,H_1$, $Tr\,H_2$, $\dots, Tr\,H_i, \dots$, by the formulas:

$$Tr\,H_1 = (\{x\}/x \in E_1)$$

$$Tr\,H_2 = Tr(H_1 \cup \{E_2\}) = Min(Tr\,H_1 \vee (\{x\}/x \in E_i))$$

$$Tr\,H_i = Min\ Tr(H_{i-1} \cup \{E_i\})$$

$$= Min(Tr\,H_{i-1} \vee (\{x\}/x \in E_i))$$

etc. ...

Finally we obtain $Tr\,H_m = Tr\,H$.

2. The coefficients τ and τ'.

For a hypergraph H we denote by $\tau(H)$ the *transversal number*, that is to say, the smallest cardinality of a transversal; similarly, we denote by $\tau'(H)$ the largest cardinality of a minimal transversal. Clearly:

$$\tau(H) = \min_{T \in Tr\,H} |T| \leq \max_{T \in Tr\,H} |T| = \tau'(H).$$

Example 1: The finite projective plane of rank r. By definition, a projective plane of rank r is a hypergraph having $r^2 - r + 1$ vertices ("points"), and $r^2 - r + 1$ edges ("lines"), satisfying the following axioms:

(1) every point belongs to exactly r lines;

(2) every line contains exactly r points;

(3) two distinct points are on one and only one line;

(4) two distinct lines have exactly one point in common.

Projective planes do not exist for every value of r (for example, if $r = 7$), but it is known that if $r = p^\alpha + 1$, with p prime, $p \geq 2$, $\alpha \geq 1$, there exists a projective plane of rank r denoted $PG(2,p^\alpha)$ built on a field of p^α elements. For example, the projective plane with seven points ("Fano configuration") is $PG(2,2)$.

It is clear that in a projective plane every line is a minimal transversal set of H. In the projective plane of seven points there are no others because $H = Tr\, H$ (given that any two edges meet and that the chromatic number of this hypergraph is > 2). For the projective planes of rank $r > 3$, we have $\tau(H) = r$, *but there exist other minimal transversals which are all of cardinality* $\geq r + 2$ (Pelikan [1971]). Hence $\tau'(H) \geq r + 2$.

On the other hand, Bruen [1971], has proved that every projective plane H of rank r satisfies $\tau'(H) \geq r + \sqrt{r-1}$.

Indeed, the minimal cardinality of a transversal T which is not a line is given by the following table for the different known projective planes of rank $r \leq 9$.

r	3	4	5	6	-	8	9
n	7	13	21	31	-	57	73
$\min \lvert T \rvert$	-	6	7	9	-	12	?

Example 2: The affine plane of rank k.

By an *affine plane* is meant the subhypergraph H of rank k obtained from a finite projective plane of rank $k+1$ by suppressing the points of a given line. Every edge of H is called a *line*, and two lines of H which have an empty intersection are said to be *parallel*.

Thus an affine plane of rank k satisfies the following properties:

Every line contains k points;

Every point belongs to $k+1$ lines;

There are k^2 points and $k^2 + k$ lines;

Two distinct points have one and only one line in common;

Two distinct lines have either no points in common ("parallel"), or a common point ("secant");

Parallelism is an equivalence relation which partitions the set of lines into $k+1$ classes of k edges each;

Through every point not belonging to a given line, there passes one and only one line parallel to the given line.

Bruen and Resmini [1983] showed that for an affine plane H of order q, we have $\tau(H) \leq 2q - 1$, and Brouwer and Schrijver [1976] showed that for the affine plane H constructed on a field of q elements, we have $\tau(H) = 2q - 1$. Finally Jamison [1977] has shown that for the hypergraph H on the vector space with a base $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ constructed on a field K of q elements and having as edges the planes $\{\Sigma x_i \mathbf{e}_i / \Sigma a_i x_i = b\}$ we have $\tau(H) = n(q-1) + 1$. This cardinality is obtained with the obvious transversal $T = \{k\mathbf{e}_i / k \in K,\ i = 1,2,\ldots,n\}$, but it is shown that we cannot do better than that.

Example 3: The (n,k,λ)-configuration. This is by definition a k-uniform hypergraph H of order n such that every pair of vertices is contained in exactly λ edges. From this definition we easily deduce that

(i) H is regular and of degree $\Delta(H) = \lambda \dfrac{n-1}{k-1}$,

(ii) H has $m(H) = \lambda \dfrac{n(n-1)}{k(k-1)}$ edges.

For certain known (n,k,λ) configurations, the transversal number τ is given by the following table.

(n,k,λ)	(13,3,1)	(10,4,2)	(9,4,3)	(11,3,3)	(12,4,3)
τ	7	4	4	7	6

Theorem 3. *Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph on X with $\tau'(H) = t$, and let k be an integer ≥ 1. If $k < |E_1| \leq |E_2| \leq \cdots \leq |E_m|$, and if every k-tuple of X is contained in at most λ edges of H, then*

$$\sum_{j=1}^{t} \binom{|E_j|-1}{k} \leq \lambda \binom{n-t}{k}$$

Proof. Let T be a minimal transversal of H. For every $x \in T$, there exists an edge E_x such that $E_x \cap T = \{x\}$. Since $E_x \neq E_y$ for $x \neq y$, the family $H' = (E_x / x \in T)$ is a partial hypergraph of H.

By counting in two different ways the pairs (A,E) where $E \in H'$ and where A is a k-tuple of $X - T$ contained in E, we obtain

(1) $$\sum_{x\in T}\binom{|E_x-\{x\}|}{k} = \sum_{\substack{A\subset X-T\\ |A|=k}} |\{E_x/E_x\supset A\}|$$

from whence, a fortiori,

$$\sum_{j=1}^{t}\binom{|E_j|-1}{k}\le\lambda\binom{n-t}{k}.$$

Corollary 1. *Let H be a hypergraph of order n with no loops, and put $s = min\,|E_i|$ and $\Delta = \Delta(H)$. Then $\tau'(H)\le\left[\frac{n\Delta}{\Delta+s-1}\right]$. Furthermore, this bound is the best possible for $s = 2$.*

Indeed, Theorem 3 with $k = 1$ gives

$$t\binom{s-1}{1}\le\Delta\binom{n-t}{1}.$$

Whence $\tau'(H) = t\le\frac{n\Delta}{\Delta+s-1}$. For $s = 2$, the equality is obtained with the Turan graph.

Corollary 2. *Let H be a linear hypergraph of order n with* $\min|E_i| = s > 2$. *Then*

$$\tau'(H)\le n+\frac{1}{2}(s^2-3s+1)-\frac{1}{2}\sqrt{4n(s^2-3s+2)+(s^2-3s+1)^2}.$$

Proof. Theorem 3 with $k = 2$ and $\lambda = 1$ gives

$$t\binom{s-1}{2}\le\binom{n-t}{2}$$

that is to say

$$t^2-t(s^2-3s+2n+1)+(n^2-n)\ge 0.$$

Equality gives a quadratic equation which has two solutions t' and t'', and we note that $t'<n<t''$. Since $\tau'(H)\le n$, we have also $\tau'(H)\le t'$. The result follows.

Corollary 3 (Erdös, Hajnal [1966]). *Let H be a linear 3-uniform hypergraph of order n; then*

$$\tau(H) \le n - \sqrt{2n + \frac{1}{4}} + \frac{1}{2}.$$

This follows from Corollary 2 with $s = 3$.

Theorem 4 (Meyer [1975]). *Let H be a hypergraph with min $|E_i| = s > 1$, and suppose that the vertices of X are labelled in such a way that*

$$d_H(x_1) \le d_H(x_2) \le \cdots \le d_H(x_n).$$

Then the number $\tau'(H) = t$ satisfies

$$\sum_{i=1}^{t} [d_H(x_i)+s-1] \le \sum_{i=1}^{n} d_H(x_i).$$

Proof. Using formula (1) of the proof of Theorem 3 with $k = 1$, we obtain

$$(1') \qquad \sum_{x \in T} (|E_x - \{x\}|) \le \sum_{x \in X-T} d_H(x).$$

This implies: $t(s-1) \le \sum_{i=t+1}^{n} d_H(x_i)$. The stated inequality follows easily.

We note that Theorem 4 generalises Corollary 1, and, in the case of graphs, generalises the theorem of Zarankiewicz (*Graphs*, chapter 13). (For an independent proof by induction, see Hansen, Lorea [1976]).

Theorem 5 (Berge, Duchet [1975]). *Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph on X. Let $\overline{E}_j = X - E_j$. We have $\tau'(H) \le k$ if and only if the hypergraph $\overline{H} = (\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m)$ is k-conformal.*

Proof. To say that $\overline{H}$ is k-conformal, is to say that for every $A \subset X$ the following two conditions are equivalent:

$$(C_k) \qquad (\forall S \subset A, |S| \le k)(\exists \overline{E}_j \in \overline{H}) : \overline{E}_j \supset S.$$

$$(C) \qquad (\exists \overline{E}_j \in \overline{H}) : \overline{E}_j \supset A.$$

Let us consider the negations of these conditions, that is:

$(\overline{C}_k)$ $\quad (\exists S \subset A, \ |S| \leq k)(\forall E_j \in H) : E_j \cap S \neq \emptyset.$

$(\overline{C})$ $\quad (\forall E_j \in H) : E_j \cap A \neq \emptyset.$

To say that $\overline{H}$ is k-conformal is to say that $(\overline{C}_k)$ is equivalent to $(\overline{C})$. On the other hand, to say that $\tau'(H) \leq k$, is equivalent to saying that every transversal A contains a transversal S with $|S| \leq k$; that is to say: $(\overline{C}) \Rightarrow (\overline{C}_k)$.

Since we have always $(\overline{C}_k) \Rightarrow (\overline{C})$, we have $\tau'(H) \leq k$ if and only if $(\overline{C}_k)$ is equivalent to $(\overline{C})$, that is to say if and only if $\overline{H}$ is a k-conformal hypergraph.

Corollary 1. *Let H be a simple hypergraph on X and let k be an integer ≥ 2. We have $\tau'(H) \leq k$ if and only if for every partial hypergraph $H' \subset H$ with $k+1$ edges there exists an edge $E \in H$ contained in the set $\{x / d_{H'}(x) > 1\}$.*

Proof. From Theorem 15 (Chapter 1), the k-conformity of $\overline{H}$ is equivalent to saying that for every $\overline{H}' \subset \overline{H}$ with $k+1$ edges, the set

$$A = \{x / x \in X, \ d_{\overline{H}'}(x) \geq k\}.$$

is contained in an edge $\overline{E}$ of $\overline{H}$. Since

$$d_{\overline{H}'}(x) = |H'| - d_{H'}(x) = (k+1) - d_{H'}(x)$$

this condition is also equivalent to:

$$\{x / x \in X, \ d_{H'}(x) \leq 1\} = A \subset \overline{E}.$$

From this the stated result follows.

Corollary 2. *Let H be a simple hypergraph with $\tau(H) = t \geq 2$. The hypergraph $Tr\,H$ is uniform if and only if for every hypergraph $H' \subset H$ of $t+1$ edges, there exists an edge $E \in H$ contained in*

$$\{x / x \in X, \ d_{H'}(x) > 1\}.$$

3. τ-critical hypergraphs

We say that a hypergraph $H = (E_1, E_2, ..., E_m)$ is *τ-critical* if the deletion of any edge decreases the transversal number, that is to say, if

$$\tau(H - E_j) < \tau(H) \qquad (j = 1,2,...,m)$$

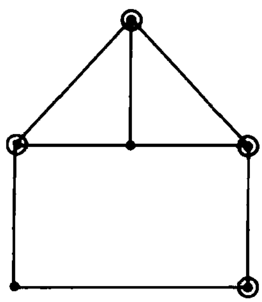
Since we cannot have $\tau(H-E_j) < \tau(H)-1$, this is equivalent to saying that if H is τ-critical with $\tau(H) = t+1$, then $\tau(H-E) = t$ for every $E \in H$.

Example 1. The hypergraph K^r_{t+r} is τ-critical, since $\tau(K^r_{t+r}) = t+1$ and if E is an edge of K^r_{t+r}, the hypergraph $K^r_{t+r} - E$ has a transversal $X - E$ of cardinality t.

Example 2. Consider the family $\mathcal{A}$ of all the $(r-1)$-tuples of a set X with $t+r-1$ elements; with every $A \in \mathcal{A}$, let us associate a new point y_A, these points forming a set Y with cardinality $\binom{t+r-1}{r-1}$. Consider the hypergraph $H = (A \cup \{y_A\} / A \in \mathcal{A})$ on $X \cup Y$. Clearly, $\tau(H) = t+1$; since $H - (A \cup \{y_A\})$ has a transversal $X - A$ of cardinality t, the hypergraph H is τ-critical.

For $r = 2$, the concept of a τ-critical graph is due to Zykov in 1949. The systematic study started in 1961 with an article by Erdős and Gallai, who showed that a τ-critical graph G without isolated vertices satisfies $2\tau(G) - n(G) \geq 0$.

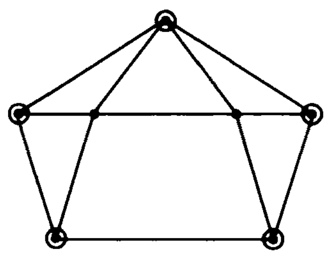
Examples of τ-critical graphs are shown in Figures 4 and 5.



$\tau = 4$

$2\tau - n = 2$

Figure 4



$\tau = 5$

$2\tau - n = 3$

Figure 5

Proposition 1. *Every τ-critical hypergraph is simple.*

For if $H = (E_1,...,E_m)$ is τ-critical and not simple, there exist two indices i and j with $E_i \subset E_j$. An optimal transversal of $H-E_j$ has $\tau(H)-1$ vertices, and since it meets E_i it also meets E_j. Therefore $\tau(H) \leq \tau(H)-1$, a contradiction.

Proposition 2. *Every hypergraph H with $\tau(H) = t+1$ has as a partial hypergraph, a τ-critical hypergraph H' with $\tau(H') = t+1$.*

Indeed, to obtain H' it is enough to remove from H as many edges as one can without changing the transversal number.

In a hypergraph H a vertex x is said to be *critical* if

(1) $\tau(H-H(x)) < \tau(H)$.

We note that (1) is equivalent to:

(2) $$\tau(H-H(x)) = \tau(H)-1.$$

Indeed, if (1) holds then the hypergraph $H_1 = H-H(x)$ has a transversal T_1 of cardinality $\tau(H)-1$. The set $T_1 \cup \{x\}$ is a transversal of H and, since its cardinality is $\tau(H)$, it is a minimum transversal. From this we obtain (2).

Conversely, if (2) holds, let T be a minimum transversal of H containing x. Then $T-\{x\}$ is a transversal of $H-H(x)$ of cardinality $\tau(H)-1$, from which (1) follows.

Proposition 3. *Every vertex of a τ-critical hypergraph is critical.*

Let H be a τ-critical hypergraph and let x be one of its vertices. Since x is contained in an edge, E say,

$$\tau(H-H(x)) \leq \tau(H-E) < \tau(H).$$

Thus x is a critical vertex.

Example 1. Let us consider a simple graph $G = (X,E)$, connected and without bridges. Let H be the hypergraph whose vertices are the edges of G and whose edges are the elementary cycles of G. Through every edge of a graph without bridges there passes a cycle; hence H is a simple hypergraph on E.

For $e_0 \in E$ there exists a maximal tree (X,F) with $e_0 \in F$ which spans G; we have $\tau(H) = m(G)-n(G)+1$, and every co-tree of G is a transversal of H. Therefore $E-F$ is a minimum transversal of H containing e_0. Thus every vertex of H is critical.

Example 2. The analogous situation holds also for a strongly connected digraph G_0. Let H be the hypergraph whose vertices are the arcs of G_0 and the edges are the elementary circuits of G_0 (for example, take G_0 to be the Möbius ladder represented in Figure 6).

Here the edges of H are:

$$E_1 = \{ab, bd, dc, ca\}$$
$$E_2 = \{ab, bf, fe, ea\}$$
$$E_3 = \{ab, bf, fe, ed, dc, ac\}$$
$$E_4 = \{ab, bd, dc, cf, fe, ea\}$$
$$E_5 = \{cf, fe, ed, dc\}$$

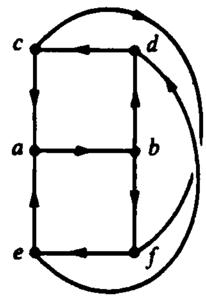


Figure 6

It is easy to see that $\tau(H) = 2$ and that every vertex of H belongs to a transversal of cardinality 2. Hence every vertex of H is critical. By way of an exercise the reader can verify Proposition 4 with this example.

Theorem 6 (Tuza [1984]). *Let $H = (E_1, E_2, ..., E_m)$ be a τ-critical hypergraph with $\tau(H) = t+1$. Then*

$$\sum_{j=1}^{m} \binom{|E_j|+t}{t}^{-1} \leq 1.$$

Proof. For every edge E_j there exists a set $T_j \in Tr(H - E_j)$ of cardinality t. Clearly $E_j \cap T_i = \emptyset$ if and only if $i = j$; thus from Theorem 6, Chapter 1,

$$\sum_{j=1}^{m} \binom{|E_j|+|T_j|}{|E_j|}^{-1} \leq 1.$$

The stated inequality follows.

Corollary 1 (Bollobas [1965]; Jaeger, Payan [1971]). *Let H be a τ-critical hypergraph of rank r, with $\tau(H) = t+1$; then the number of its edges satisfies:*

$$m(H) \leq \binom{r+t}{t}.$$

Moreover this bound is attained with the hypergraph K_{t+r}^r.

Proof. Let $E \in H$. Since $|E| \leq r$ we have

$$\binom{|E|+t}{t} \leq \binom{r+t}{t}.$$

Thus

$$m(H)\binom{r+t}{t}^{-1} \leq \sum_{E \in H} \binom{|E|+t}{t}^{-1} \leq 1.$$

The stated inequality follows.

We verify immediately that equality holds for $H = K_{t+r}^r$.

Corollary 2 (Theorem of Erdős, Hajnal and Moore). *If G is a simple graph of order n with $\alpha(G) = k$ and $\alpha(G-E_j) = k+1$ for every edge E_j, then*

$$m(G) \leq \binom{n-k+1}{2}.$$

Since every maximum stable set of G is the complement of a minimal transversal G and vice versa, we have $\tau(G) = n-k$, $\tau(G-E_j) = n-k-1$ for every j. The stated inequality then follows from Corollary 1.

The following result is a theorem of Gyarfas, Lehel, Tuza [1980], which extends a theorem of Hajnal (*Graphs*, Theorem 8, Chapter 13).

Theorem 7. *Let H be a τ-critical hypergraph on X with $\tau(H) = t+1$. Let $\mathcal{A}$ be the set of subsets A of X such that $A \notin H$ and $A \cup \{x\} \in H$ for some $x \in X$. For $x \in X$ and $Y \subset X$, put*

$$\Gamma x = \{A / A \in \mathcal{A}, A \cup \{x\} \in H\}$$

$$\Gamma Y = \bigcup_{x \in Y} \Gamma x$$

Then every set $S \subset X$ such that $|S \cap E| \leq 1$ for all $E \in H$ satisfies $|\Gamma S| \geq |S|$.

(*) **Proof.** Let $\mathcal{S}$ be the family of $S \subset X$ such that $|S \cap E| \leq 1$ for every $E \in H$. We shall suppose that there exists in $\mathcal{S}$ a set S which satisfies $|\Gamma S| < |S|$, and which is minimal with respect to this property. We shall then deduce a contradiction.

From the König-Hall theorem (*Graphs*, Theorem 5, Chapter 7) this means that the bipartite graph $G = (X, \mathcal{A}; \Gamma)$ has no matching of S into $\mathcal{A}$, but for every $y \in S$ there exists a matching of $S - \{y\}$ into $\mathcal{A}$. Since in G the degree of a point of X is ≥ 1, and since $|\Gamma S| < |S|$, there exists a set $A_0 \in \Gamma S$ and two distinct points $y_1, y_2 \in S$ such that

$$A_0 \cup \{y_1\} = E_1 \in H, \qquad y_1 \in S$$

$$A_0 \cup \{y_2\} = E_2 \in H, \qquad y_2 \in S.$$

Since $\tau(H - E_1) = t$, let T_1 be a transversal set of the hypergraph $H - E_1$ having cardinality t. Since $T_1 \cap E_1 = \varnothing$, we have $y_1 \notin T_1$, and consequently

$$T_1 \cap A \neq \varnothing \qquad (A \in \Gamma y_1,\ A \neq A_0)$$

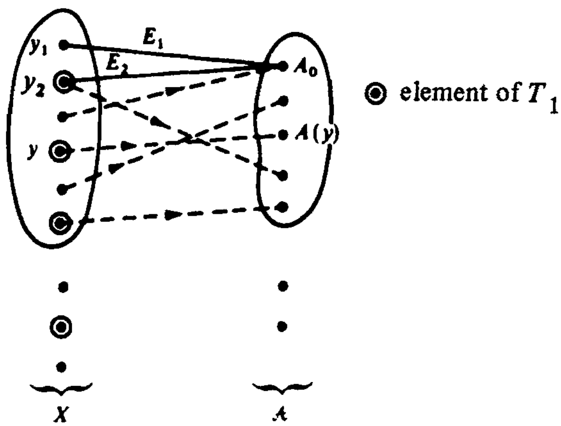


Figure 7

Because of the minimality of S, we have $|\Gamma Y| \geq |Y|$ for every $Y \subset S - \{y_1\}$, and hence there exists a matching of $S - \{y_1\}$ into ΓS. This matching makes correspond to every $y \in S - \{y_1\}$ a set $A(y) \in \Gamma y$; and, since $|\Gamma S| = |S| - 1$, every $A \in \Gamma S$ is the image of some $y \in S - \{y_1\}$.

Consider a set T_2 obtained from T_1 by replacing every vertex $y \in S - \{y_1\}$ which belongs to T_1 by a vertex chosen arbitrarily from the set $A(y)$.

We note that if an $A \in \Gamma S$ satisfies $T_1 \cap A = \emptyset$, then all the points $y \in S - \{y_1\}$ joined to A in G are elements of T_1. Hence

$$T_2 \cap A \neq \emptyset \quad (A \in \Gamma S).$$

Since $S \in \boldsymbol{S}$ this implies that

$$T_2 \cap E \neq \emptyset \quad (E \in H, E \cap S \neq \emptyset).$$

It follows that T_2 is a transversal of H, and since $|T_2| \leq |T_1| = t$ we have a contradiction.

4. The König property

A *matching* in a hypergraph H is a family of pairwise disjoint edges, and the maximum cardinality of a matching is denoted $\nu(H)$.

A matching can also be defined as a partial hypergraph H_0 with $\Delta(H_0) = 1$.

We note that for every transversal T and for every matching H_0,

$$|T \cap E| \geq 1 \quad (E \in H_0)$$

Thus $|H_0| \leq |T|$, from whence

$$\nu(H) = \max |H_0| \leq \tau(H).$$

We say that H has the *König property* if $\nu(H) = \tau(H)$.

A *covering* of H will be a family of edges which covers all the vertices of H, that is to say a partial hypergraph H_1 with $\delta(H_1) = \min_{x \in X} d_{H_1}(x) \geq 1$. We write

$$\rho(H) = \min |H_1|.$$

Finally, a *strongly stable* set of H is by definition a set $S \subset X$ such that $|S \cap E_1| \leq 1$ for every $E \in H$, and we write

$$\overline{\alpha}(H) = max\, |S|.$$

It is seen immediately that $\rho(H) = \tau(H^*)$, $\overline{\alpha}(H) = \nu(H^*)$; for this reason we say that H has the *dual König property* if $\rho(H) = \overline{\alpha}(H)$.

Example 1: The r-partite complete hypergraph. If $n_1 \leq n_2 \leq \cdots \leq n_r$, the hypergraph $K^r_{n_1, n_2, \ldots, n_r}$ has the König property since $\tau = n_1$ and $\nu = n_1$. It also has the dual König property since $\rho = n_r$ and $\overline{\alpha} = n_r$.

Example 2: Semi-convex polyominoes. A *polyomino* P is a finite set of unit squares in the plane arranged like a chessboard with some of its squares cut out. With every polyomino P one can associate a hypergraph whose vertices are the unit squares of P and whose edges are the maximal rectangles contained in P.

It is easy to see that this hypergraph P has the Helly property and is conformal.

Moreover, if P is "semi-convex", that is to say if every horizontal line of the plane intersects P in an interval, the hypergraph P has the König property (Berge, Chen, Chvatal, Seow [1981]) and the dual König property (Györi [1984]). The smallest polyomino P with $\nu(P) \neq \tau(P)$ is shown in Figure 8.

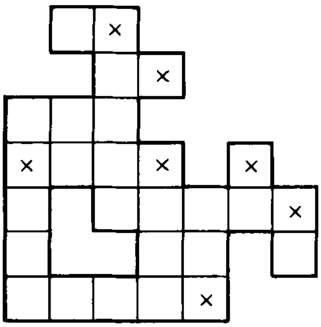


Figure 8.
Polyomino with $\nu = 6$ and $\tau = 7$.

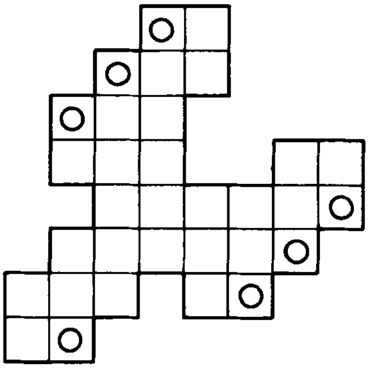


Figure 9.
Polyomino with $\rho = 8$ and $\bar{\alpha} = 7$.

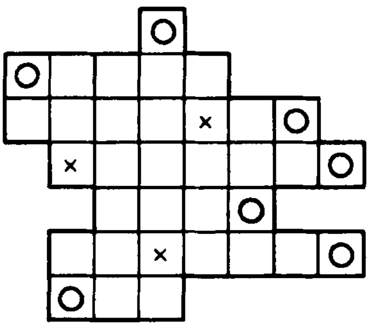


Figure 10. *Semi-convex polyomino with* $\nu = \tau = 3$, $\rho = \bar{\alpha} = 7$.

Example 3: Paving with bricks. Consider the integers $a \leq b$, $p \leq q$, and a rectangular chessboard of dimensions $p \times q$, which is to be paved with bricks of dimensions $a \times b$. What is the maximum number of bricks that one can place on the chessboard?

We can consider the hypergraph H whose vertices are the unit squares and whose edges are all the rectangles of dimension $a \times b$; the answer to the problem is then $\nu(H)$. Brualdi and Foregger [1974] have proved that H has the König property for every (p,q) if and only if a is a divisor of b. For example, for $a = 2$, $b = 3$, there exists a chessboard of dimensions 9×6 which determines a hypergraph H with $\nu(H) = 9$, $\tau(H) = 10$, thus not satisfying the König property (Figure 11).

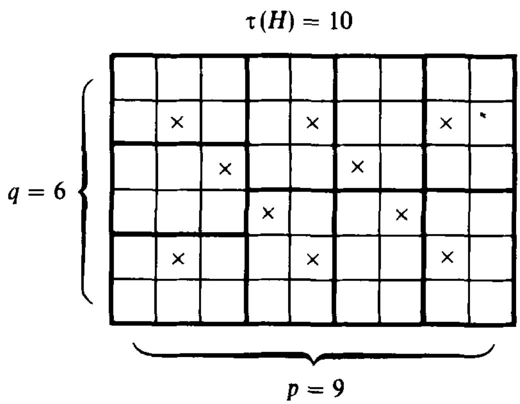


Figure 11. *The squares marked with a cross represent an optimal transversal of H.*

If one wishes to pave with bricks of dimension $a \times b$ a "truncated" chessboard, we have, in general, neither the König property nor the dual König property; nevertheless, the truncated chessboard of 24 squares represented in Figures 12 and 13 satisfies these two properties with bricks of dimensions 1×4, as the reader can easily verify.

$\tau(H) = 5$

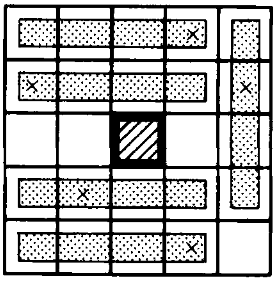


Figure 12.
The squares marked with a cross constitute a transversal of H and consequently this matching is is optimal.

$\bar{\alpha}(H) = 8$

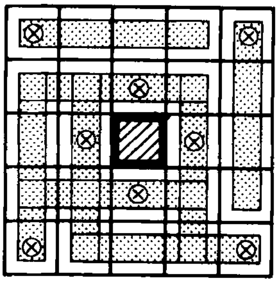


Figure 13.
The squares marked with a circle constitute a strongly stable set and consequently this covering is optimal.

Example 4: Hypergraph of subtrees of a tree. Let G be a tree on $X = \{x_1, x_2, \ldots, x_n\}$, and let $H = (E_1, E_2, \ldots, E_m)$ be a family of subsets of X which induce a subtree. We have seen that H has the Helly property. It follows from the theory of perfect graphs that H also has the König property.

Let us give a proof by induction on $\tau(H) = t$ for the equality $\nu = \tau$. If $t = 1$, it is clear that $\nu = \tau$. So, we may assume that H has an optimal transversal $T = \{x_1, x_2, \ldots, x_t\}$ with $t \geq 2$.

Let $S \subset X$ be a minimal set such that the subgraph G_S is a tree containing T. Furthermore, let us choose T such that $|S|$ is minimum. A pendent vertex x_1 of the tree G_S is therefore in T.

Since T is a minimal transversal of H, the partial hypergraph $H_1 = (E / E \in H, E \cap T = \{x_1\})$ is non-empty; there exists an edge $E_1 \in H_1$ such that $E_1 \cap (S - \{x_1\}) = \emptyset$ (by the minimality of $|S|$).

The hypergraph $H' = H - H(x_1)$ has a transversal of cardinality $t-1$. Thus $\nu(H') = t-1$ (by the induction hypothesis). An optimal matching of H' augmented by the edge E_1, forms a matching of H with cardinality t, and hence $\nu(H) \geq t = \tau(H)$. We therefore have $\nu(H) = \tau(H)$.

Example 5: Bipartite multigraphs. A famous theorem of König states that a bipartite multigraph has the König property, and also the dual König property.

For non-bipartite graphs, those having the König property have been characterised by Sterboul and this result will be proved later on (Chapter 4, Theorem 6).

Example 6: Interval hypergraphs. A theorem of Gallai states that an interval hypergraph has the König property. This follows also from Example 2 or Example 4 above. We shall see later on that it also has the dual König property.

Example 7: The hypergraph of circuits of a digraph. Let G_0 be a strongly connected digraph, and let H be the hypergraph whose vertices are the arcs of G_0 and whose edges are the elementary circuits of G_0.

If G_0 is planar, a theorem of Lucchesi and Younger in [1978] shows that the hypergraph H has the König property. If G_0 is non-planar, the hypergraph H does not in general have the König property: for the graph G_0 of Figure 6 we find that $\nu(H) = 1$ and $\tau(H) = 2$. Younger has also conjectured that if G_0 is planar, the hypergraph $Tr\,H$ has the König property; Kahn [1984] has shown that for G_0 planar the hypergraph H_1 of minimal length circuits of G_0 has its transversal hypergraph $Tr\,H_1$ with the König property.

Theorem 8 (Seymour [1982]). *A linear hypergraph H with $n(H)$ vertices and $m(H)$ edges without repeated loops satisfies*

$$\nu(H) \geq \frac{m(H)}{n(H)}.$$

(*) Proof. Let H be a linear hypergraph with $m(H) = m$, $n(H) = n$. Let $p(H) = p$ be the least integer $\geq \frac{m}{n}$. Thus

(1) $$p \geq \frac{m}{n}$$

(2) $$p-1 < \frac{m}{n}.$$

We show that $\nu(H) \geq p$. As this is trivial for $p = 1$, we may assume that $p(H) \geq 2$ and prove the result by induction on m.

1. We can suppose that for every $E \in H$, there are at least $(p-2)|E|+n+1$ edges of H which meet E.

For if not, the hypergraph $H_1 = (F/F \in H, F \cap E = \emptyset)$ satisfies

$$m - m(H_1) < (p-2)|E|+n+1$$

Hence, from (2),

$$m(H_1) > n(p-1)+1-(p-2)|E|-n-1 = (n-|E|)(p-2).$$

In this case

$$\frac{m(H_1)}{n(H_1)} \geq \frac{(n-|E|)(p-2)}{n-|E|} = p-2.$$

By virtue of the induction hypothesis the hypergraph H_1, which is linear, satisfies $\nu(H_1) \geq p-1$. By adjoining E to a matching of H_1 with $p-1$ edges we obtain a matching of H with p edges, and the theorem is proved.

2. *If $S \subset X$, $|S| \leq p-1$, there exists an edge $E \in H$ with $E \cap S = \emptyset$.*

Let $x \in X$. The sets $E-\{x\}$ with $E \in H(x)$ are pairwise disjoint (by the linearity of H); since their union has at most $n-1$ points, and only one of them can be empty, we have $|H(x)| \leq n$. Thus the maximum degree of H is $\Delta(H) \leq n$.

Using (2) we see that the partial hypergraph $H' = (E/E \in H, E \cap S \neq \emptyset)$ satisfies

$$m(H') \leq |S|\Delta(H) \leq (p-1)n < m = m(H)$$

Thus there is an edge $E \in H - H'$, and $E \cap S = \emptyset$.

3. We shall define progressively distinct edges $F_1, F_2, \dots, F_p$ and distinct vertices $x_1, x_2, \dots, x_p$ by the following rules:

(I) F_1 is an edge of maximum cardinality; x_1 is a point of F_1 with the smallest degree.

(II) For $i > 1$, F_i is an edge such that $F_i \cap \{x_1, x_2, \dots x_{i-1}\} = \emptyset$ with the smallest cardinality (from assertion 2 above such an edge exists); x_i is a vertex of F_i for which $d_H(x)$ is maximum.

Put

$$|F_i| = f_i$$

$$H_i = (E/E \in H, E \cap \{x_1, x_2, \ldots, x_i\} = \{x_i\})$$

$$H_i^0 = (E/E \in H_i, |E| = f_i)$$

We note that $f_1 \leq f_2 \leq \cdots \leq f_p$ and that $F_1 \in H_i^0$. We show that

(3) $$|H_i^0| \geq f_i\, |H_i| - n + 1 + \sum_{j \in J_i} (f_j - 1)$$

where

$$J_i = \{j/1 \leq j < i;\ x_i \in \bigcup_{E \in H_i} E\}.$$

We note that if $j \in J_i$ there exists a unique edge $E \in H_j$ which satisfies $x_i \in E$ and that this edge E has at least f_j elements. Thus

$$n-1 \geq \sum_{x_i \in E} (|E|-1) = \sum_{E \in H_i} (|E|-1) + \sum_{j \leq i} \sum_{\substack{E \in H_j \\ x_i \in E}} (|E|-1)$$

$$\geq \sum_{E \in H_i} f_i - \sum_{E \in H_i^0} 1 + \sum_{j \in J_i} (f_j - 1)$$

$$= f_i\, |H_i| - |H_i^0| + \sum_{j \in J_i} (f_j - 1)$$

From this (3) follows.

4. We show

(4) $$f_i\, |H_i| - n \geq f_i (p - 1 - |J_i|).$$

From assertion 1 above the number of edges of H which meet F_i is at least $(p-2)f_i + n + 1$. For $x \in F_i$,

$$|H(x)| \leq |H(x_i)|.$$

Thus

$$(p-2)f_i + n + 1 \leq f_i(|H_i| + |J_i| - 1) + 1.$$

From this (4) follows.

5. We show that

(5) $$|H_i^0| > p-i+\sum_{j<i}(f_j-1)$$

From (3) and (4) we obtain

$$|H_i^0| \geq 1+f_i(p-1-|J_i|)+\sum_{j\in J_i}(f_j-1)$$

$$= 1+f_i(p-1-|J_i|)-\sum_{\substack{j<i\\ j\notin J_j}}(f_j-1)+\sum_{j<i}(f_j-1)$$

However,

$$f_i(p-1-|J_i|)-\sum_{\substack{j<i\\ j\notin J_i}}(f_j-1) \geq f_i(p-1-|J_i|)-\Sigma f_j$$

$$=f_i(p-1-|J_i|)-f_i(i-1-|J_i|) = f_i(p-1) \geq p-i.$$

From this (5) follows.

6. We shall define a sequence of edges $E_1,E_2,...,E_p$ one by one; if $E_1,E_2, \ldots ,E_{i-1}$ have been defined, we take $E_i \in H_i^0$ so that $E_i \cap Z_i = \emptyset$, where

$$Z_i = \{x_1,x_2,...,x_{i-1}\} \cup \bigcup_{j<i}(E_j-\{x_j\})$$

Such an edge E_i exists, for the sets $(E-\{x_i\}/E \in H_i^0)$ are pairwise disjoint and there are at least $1+|Z_i|$ of them from (5); thus at least one of them is disjoint from Z_i.

Every edge E_j with $j<i$ is disjoint from the edge E_i since $x_j \notin E_i$ (because $E_i \in H_i^0 \subset H_i$), and $(E_j-\{x_j\}) \cap E_i \subset Z_i \cap E_i = \emptyset$.

Thus $(E_1,E_2,...,E_p)$ is a matching, and hence $\nu(H) \leq p$.

Q.E.D.

Corollary (Theorem of DeBruijn and Erdős, completed by Ryser [1970]). *Let* $H = (E_1,E_2,...,E_m)$ *be a family of distinct subsets of* X, *where* $|X| = n$, *such that* $|E_i \cap E_j| = 1$ *for* $i \neq j$. *Then* $m \leq n$. *Furthermore, if* $m = n$, *we have one of the following cases:*

(i) *H is a projective plane of rank* $r \geq 3$;

(ii) $H = (\{1\},\{1,2\},\{1,3\},...,\{1,n\}), \quad n \geq 1;$

(iii) $\qquad H = (\{1,2\},\{1,3\}, \ldots ,\{1,n\},\{2,3,\ldots,n\}), \quad n \geq 3.$

Inequality $m \leq n$ is obvious since, from Theorem 8,

$$\nu(H) = 1 \geq \frac{m}{n}$$

We note that by using this result, Seymour has also shown that if H is a linear hypergraph H and satisfies $\nu(H) = \frac{m}{n}$, then we have either (i), (ii), (iii) or

(iv) $\qquad H = K_n$, where n is odd and ≥ 5.

Exercises on Chapter 2.

Exercise 1 (§2)

Show that if H has the Helly property and if we put

$$H_i = \{E/E \in H, E \subset X - E_i\}$$

then $\tau(H) \leq \max_i m(H_i)$.

Exercise 2 (§2)

Let H be an r-uniform hypergraph of maximum degree $\Delta = 2$. The upper bound for $\tau(H)$ has been determined by Sterboul [1970]:

if r is even, it is $\left[\left[\frac{2n}{r}\right]\frac{2}{3}\right]$;

if r is odd, it is $\left[\frac{4n}{3r+1}\right]$ or $\left[\frac{4n}{3r+1}\right]$.

Try to construct hypergraphs for which this bound is obtained.

Exercise 3 (§2)

If H is a 3-uniform regular of degree $\Delta = 3$, then

$$\tau(H) \leq \left[\frac{n}{2}\right].$$

Show that this bound is the best possible. (Henderson, Dean [1974]).

Exercise 4 (§2) Let H be a hypergraph without loops on X. For every $Y \subset X$, define

$$H/Y = (E_i / E_i \in H,\ \ E_i \subset Y).$$

Put $\tau(H) = 0$ if H is "empty" (having no edges), and suppose that

$$\tau(H/Y) \leq \frac{|Y|}{2} \qquad (Y \subset X)$$

Show that for every maximal transversal $T = \{x_1, x_2, \ldots, x_t\}$, there exist distinct elements $y_1, y_2, \ldots, y_t$ of $S = X - T$ such that $[x_1, y_1], [x_2, y_2], \ldots, [x_t, y_t]$ are the edges of the graph $[H]_2$. (Lehel [1982]).

Hint: Consider the bipartite graph $G = (T, S; \Gamma)$ formed by the edges of $[H]_2$. The partial hypergraph $H_1 = (E_i / E_i \in H, E_i \subset A \cup \Gamma_G A)$ has a transversal T_1 with

$$|T_1| \leq \frac{1}{2} |A \cup \Gamma_G A|.$$

$T_0 = T_1 \cup (T - A)$ is a transversal of H and $|T_0| \geq |T|$ implies that $|\Gamma_G A| \geq |A|$, from which the theorem follows.

Exercise 5 (§4)

Show that the hypergraph P defined by a polyomino (Example 2, § 4) is conformal. Show that there exists a vertex of degree 1. Show that there exist distinct vertices $x_1, x_2, \ldots, x_m$ such that $x_i \in E_i$ for $i = 1, 2, \ldots, m$.

Exercise 6 (§4)

Show that the hypergraph P defined by a semi-convex polyomino (Example 2, § 4) has a set $S \subset X$ which is a transversal and is strongly stable.

Exercise 7 (§4)

Use the results of Seymour to prove the "friendship theorem" (Erdös): if in a set of n individuals, any two of them have exactly one friend in common, then there exists someone who is a friend of all the others.

Chapter 3

Fractional Transversals

1. Fractional transversal number

Let s be a positive integer. An *s-matching* of a hypergraph H on X is a function q on the edges of H such that for each edge E, $q(E) \in \{0,1,2,\ldots,s\}$, and for each vertex x,

$$\sum_{E \in H(x)} q(E) \leq s.$$

The *value* of an s-matching is $\sum_{E \in H} q(E)$; we denote by $\nu_s(H) = \max_q \sum_{E \in H} q(E)$ the maximum value of the s-matchings of the hypergraph H. Clearly, for $s = 1$, an s-matching is a matching and $\nu_1(H) = \nu(H)$.

A *fractional matching* is a real-valued function q such that

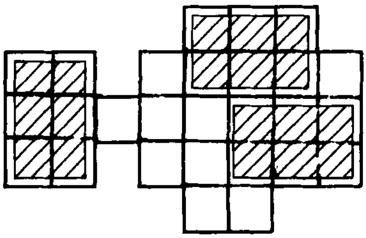
$$(1) \quad 0 \leq q(E) \leq 1 \qquad (E \in H)$$

$$(2) \quad \sum_{E \in H(x)} q(E) \leq 1 \qquad (x \in X)$$

We denote the *maximum value of a fractional matching* of H by:

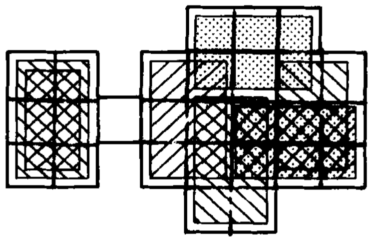
$$\nu^*(H) = \max_q \sum_{E \in H} q(E).$$

Example: Consider a truncated checkerboard, for example that of Figure 1 which has 27 squares. We wish to place a number of rectangular cards of dimension 2×3 on the board so that each card covers exactly 6 squares (or "polyominoes" of shape 2×3). What is the maximum number of polyominoes which we may place on the board so that no two of them overlap? If we let H be the hypergraph on the set of squares of the board whose edges are the sets of squares which may be covered by a polyomino, the answer is $\nu(H)$. Here $\nu(H) = 3$, and a matching of value 3 is shown in Figure 1. More difficult is the following problem: What is the maximum number of polyominos which may be placed on the board in such a way that no square is covered more than twice? The answer is $\nu_2(H)$. Here $\nu_2(H) = 7$, and a 2-matching of value 7 is shown in Figure 2. A more detailed study shows further that $\nu^*(H) = \dfrac{7}{2}$.



$\nu(H) = 3$

Figure 1



$\nu_2(H) = 7$

Figure 2

For an integer $k \geq 1$ we define a *k—transversal* of H to be a function p on X such that for each vertex x, $p(x) \in \{0,1,2,...,k\}$ (the "weights") and

$$\sum_{x \in E} p(x) \geq k \qquad (E \in H).$$

The *value* of a k-transversal p is $\sum_{x \in X} p(x)$, and we shall denote by $\tau_k(H)$ the minimum value of the k-transversals of H. Clearly, for $k = 1$ a k-transversal is a transversal and $\tau_1(H) = \tau(H)$.

A *fractional transversal* of H is a real function $p(x)$ such that

(1) $0 \leq p(x) \leq 1 \qquad (x \in X)$

(2) $\sum_{x \in E} p(x) \geq 1 \qquad (E \in H)$

The *fractional transversal number* of H is the minimum value $\tau^*(H)$ of the fractional transversals of H; this number will be our principal subject of study in this chapter.

Example. If H is the graph C_5 (a cycle of length 5) we see immediately that $p(x) \equiv 1$ is a 2-transversal, and $\tau_2(H) = 5$. Further $p(x) \equiv 0.5$ is a fractional transversal, and $\tau^*(H) = 2.5$. Further, $\nu_1(H) = 2$, $\nu_2(H) = 5$, $\nu^*(H) = 2.5$.

Remark: Let $H = (E_1,E_2,...,E_m)$ be a hypergraph on $X = \{x_1,x_2,...,x_n\}$ and let $A = ((a_j^i))$ be the incidence matrix of H:

$$a_j^i = \begin{cases} 0 \text{ if } x_i \notin E_j \\ 1 \text{ if } x_i \in E_j \end{cases}$$

A fractional matching may be interpreted as a vector $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ of the polyhedron:

$$Q = \{\mathbf{q}/\mathbf{q} \in \mathbb{R}^m,\ \mathbf{q} \geq \mathbf{0},\ A\mathbf{q} \leq \mathbf{1}\}$$

This polyhedron in m-dimensional space is thus called the *matching polytope* of H, and a matching is a vector of Q whose coordinates are either 0 or 1. Similarly, a fractional transversal may be interpreted as a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of the polytope:

$$P = \{\mathbf{p}/\mathbf{p} \in \mathbb{R}^n,\ \mathbf{p} \geq \mathbf{0},\ A^*\mathbf{p} \geq \mathbf{1}\}.$$

This polyhedron is called the *transversal polytope.*

Theorem 1 (Berge, Lovász, Simonovits). *Every hypergraph H satisfies:*

$$\nu(H) = \min_{s \geq 1} \frac{v_s(H)}{s} \leq \max_{H' \subseteq H} \frac{m(H')}{\Delta(H')} \leq \max_{s \geq 1} \frac{\nu_s(H)}{s} = \nu^*(H)$$

$$= \tau^*(H) = \min_{k \geq 1} \frac{\tau_k(H)}{k} \leq \min_{A \subseteq X} \frac{|A|}{s(H_A)} \leq \max_{k \geq 1} \frac{\tau_k(H)}{k} = \tau(H).$$

These inequalities are called the "fundamental inequalities"; the expressions "max" (or "min") imply that the upper (or lower) bound is attained.

Proof. 1. $\nu(H) = \min \dfrac{\nu_s(H)}{s}$.

If H' is a matching of size $\nu(H)$, the hypergraph sH' obtained from H by repeating each edge s times is an s-matching: thus $\nu_s(H) \geq s\nu(H)$. The equality is satisfied for $s = 1$, so indeed $\min \dfrac{\nu_s(H)}{s} = \nu(H)$.

2. $\min \dfrac{\nu_s(H)}{s} \leq \max \dfrac{m(H')}{\Delta(H')}$.

Let H'' be a maximum matching of H; we have

$$\nu(H) = \frac{m(H'')}{\Delta(H'')} \leq \max_{H' \subseteq H} \frac{m(H')}{\Delta(H')}$$

3. $\max \dfrac{m(H')}{\Delta(H')} \leq sup \dfrac{\nu_s(H)}{s}$.

Let $H'' \subset H$ be such that $\dfrac{m(H'')}{\Delta(H'')} = \max\limits_{H' \subseteq H} \dfrac{m(H')}{\Delta(H')}$. If we set $s = \Delta(H'')$, then

$$\max \frac{m(H')}{\Delta(H')} = \frac{m(H'')}{\Delta(H'')} \leq \frac{\nu_s(H)}{s} \leq \sup_s \frac{\nu_s(H)}{s}$$

4. $\sup\limits_s \dfrac{\nu_s(H)}{s} = \max \dfrac{\nu_s(H)}{s} = \nu(H)$.

Let $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ be a maximum s-matching of H. Since $\dfrac{\mathbf{z}}{s} \in Q$, we have

$$(1) \quad \frac{\nu_s(H)}{s} = \Sigma \frac{z_i}{s} \leq \nu^*(H).$$

Conversely, let q be a fractional matching with $\Sigma\, q_i = \nu^*(H)$. Since such a q will be an extremal point of a polyhedron Q defined by linear inequalities with integer coefficients, we may assume that the q_i's are rational. Let $\mathbf{z}$ be a vector such that

$$q_i = \frac{z_i}{s};\ s, z_1, z_2, \ldots, z_m \text{ integers} \geq 0$$

Since $\mathbf{z} = (z_1, z_2, \ldots, z_m) \geq 0$ and $A\mathbf{z} = A(s\mathbf{q}) = sA\mathbf{q} \leq s.1$, the vector $\mathbf{z}$ is an s-matching, whence

$$\nu^*(H) = \frac{1}{s} \sum_{i=1}^{m} z_i \leq \frac{\nu_s(H)}{s}$$

Consequently, from (1), $\dfrac{\nu_s(H)}{s} = \nu^*(H)$, and

$$\sup \frac{\nu_s(H)}{s} = \max \frac{\nu_s(H)}{s} = \nu^*(H).$$

5. $\nu^*(H) = \tau^*(H)$.

This is an immediate result of the duality theorem in linear programming: $\min\limits_{p \in P} \Sigma\, p_i = \max\limits_{q \in Q} \Sigma\, q_j$.

6. $\tau^*(H) = \min \dfrac{\tau_k(H)}{k} = \inf \dfrac{\tau_k(H)}{k}$.

Let $\mathbf{p}$ be a fractional transversal with $\Sigma\, p_i = \tau^*(H)$. We may assume that the coordinates of $\mathbf{p}$ are rational (since the extremal points of the polyhedron P have rational coordinates). Let $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ be such that

$$p_i = \frac{t_i}{k}\ ;\ t_1, t_2, \ldots, t_n \text{ integers} \geq 0.$$

Since $A^*\mathbf{p} \geq 1$ we have $A^*\mathbf{t} \geq k$: thus $\mathbf{t}$ is a k-transversal, whence

$$\tau^*(H) = \frac{\Sigma t_i}{k} \geq \frac{\tau_k(H)}{k}.$$

Conversely, every integer k satisfies $\frac{\tau_k(H)}{k} \geq \tau^*(H)$ and consequently

$$\inf_k \frac{\tau_k(H)}{k} = \min_k \frac{\tau_k(H)}{k} = \tau^*(H).$$

7. $\min \frac{\tau_k(H)}{k} \leq \min \frac{|A|}{s(H_A)}$.

Let A be a set of vertices of H. Put

$$s = s(H_A) = \min_i |E_i \cap A|.$$

Then the characteristic function of the set A is an s-transversal, whence $\tau_s(H) \leq |A|$ and consequently

$$\min_k \frac{\tau_k(H)}{k} \leq \frac{\tau_s(H)}{s} \leq \frac{|A|}{s(H_A)}.$$

Since this is true for all $A \subseteq X$,

$$\min_k \frac{\tau_k(H)}{k} \leq \min_A \frac{|A|}{s(H_A)}.$$

8. $\min \frac{|A|}{s(H_A)} \leq \max \frac{\tau_k(H)}{k}$.

Let T be a minimum transversal of H; we have

$$\min_A \frac{|A|}{s(H_A)} \leq \frac{|T|}{s(H_T)} = |T| = \frac{\tau_1(H)}{1} \leq \max_k \frac{\tau_k(H)}{k}.$$

9. $\max \frac{\tau_k(H)}{k} = \tau(H)$.

Let T be a minimum transversal and let $t(x)$ be its characteristic function:

$$t(x) = \begin{cases} 1 \text{ if } x \in T \\ 0 \text{ if } x \notin T \end{cases}$$

For each integer k, the function $kt(x)$ is a k-transversal; thus

$$\tau_k(H) \leq \sum_{x \in X} kt(x) = k\,|T|$$

whence:

$$\max_k \frac{\tau_k(H)}{k} \leq |T| = \tau(H).$$

Corollary 1. *A hypergraph H with the Kőnig property contains k disjoint edges if the only if*

$$ks(H_A) \leq |A| \qquad (A \subset X)$$

Indeed, for a hypergraph H satisfying $\nu(H) = \tau(H)$ *we have* $\nu(H) = \min_{A \subseteq X} \frac{|A|}{s(H_A)}$. *Hence* $\nu(H) \geq k$, *which is equivalent to the condition stated.*

Corollary 2. *A hypergraph H having the Kőnig property contains a set of k vertices which meet every edge if and only if*

$$k\Delta(H') \geq m(H') \qquad (H' \subset H).$$

(Similar proof).

Corollary 3. *Every r-uniform regular hypergraph has* $p(x) \equiv \frac{1}{r}$ *as an optimal fractional transversal.*

Indeed, consider a regular r-uniform hypergraph H of order n. By counting the number of edges in the bipartite edge-vertex incidence graph in two different ways we see that $m(H)r = \Delta(H)n$. Thus, from Theorem 1,

$$\frac{n}{r} = \frac{m(H)}{\Delta(H)} \leq \max_{H'} \frac{m(H')}{\Delta(H')} \leq \tau^*(H) \leq \min_A \frac{|A|}{s(H_A)} \leq \frac{n}{r}.$$

Thus $\tau^*(H) = \frac{n}{r}$ and consequently $p(x) \equiv \frac{1}{r}$ is optimal.

For example, for the complete hypergraph K_n^r, Corollary 3 gives

$$\tau^*(K_n^r) = \frac{n}{r},$$

and we have

$$\nu(K_n^r) = [\frac{n}{r}] \leq \tau^*(K_n^r) = \frac{n}{r} \leq \tau(K_n^r) = n-r+1.$$

As another example, for the cycle C_5, Corollary 3 gives

$$\tau^*(C_5) = \frac{5}{2},$$

and we have

$$\nu(C_5) = 2 \leq \tau^*(C_5) = \frac{5}{2} = \tau(C_5) = 3.$$

Theorem 1 may equally well be applied to the dual hypergraph H^*; it then has a totally different interpretation.

For an integer $k \geq 1$, a *strongly k-stable* function is a function f which assigns to each vertex x of H an integer $f(x) \in \{0,1,2,...,k\}$ such that

$$\sum_{x \in E} f(x) \leq k \qquad (E \in H)$$

We denote by $\overline{\alpha}_k(H)$ the maximum value of $\sum_{x \in X} f(x)$ for the strongly k-stable functions of H. It is clear that, for $k = 1$, a strongly k-stable function may be identified with a stable set, and $\overline{\alpha}_1(H) = \overline{\alpha}(H)$.

Proposition 1. *If H^* is the dual of H then*

$$\overline{\alpha}_k(H) = \nu_k(H^*).$$

Indeed, a strongly k-stable function on H defines a k-matching of H^*, and vice versa.

Proposition 2. *Let H be an r-uniform hypergraph of order n, and let λ,k,k' be integers with $k+k' = \lambda r$. Then*

$$\overline{\alpha}_k(H) = \lambda n = \tau_{k'}(H).$$

Indeed, f is a k-stable function if

$$\sum_{x \in E} f(x) \leq k \qquad (E \in H)$$

This is equivalent to saying that the function $p(x) = \lambda - f(x)$ satisfies

$$\sum_{x \in E} p(x) = \lambda r - \sum_{x \in E} f(x) \geq \lambda r - k = k'.$$

This means that p is a k'-transversal of H. Further

$$\sum_{x \in X} f(x) = \lambda n - \sum_{x \in X} p(x)$$

Thus

$$\bar{\alpha}_k(H) = \max \sum_{x \in X} f(x) = \lambda n - \min \sum_{x \in X} p(x) = \lambda n - \tau_{k'}(H).$$

Observe that if the hypergraph is a graph G, we may set $\lambda = k = k' = 1$ to obtain the well known equality

$$\alpha(G) + \tau(G) = n.$$

For an integer $s \geq 1$, an *s-covering* of H is a function g which assigns to each edge E an integer $g(E) \in \{0,1,2,...,s\}$ such that

$$\sum_{E \in H(x)} g(E) \geq s \qquad (x \in X).$$

We denote by $\rho_s(H)$ the minimum value of an s-covering of H.

Proposition 3. *If H^* is the dual of the hypergraph H, then*

$$\rho_k(H) = \tau_k(H^*).$$

Indeed, an s-covering of H corresponds in H^* to an s-transversal, and vice-versa.

Proposition 4. *Let H be a regular hypergraph with $\Delta(H) = h$, and let λ, s, t be integers such that $s+t = \lambda h$. Then*

$$\rho_s(H) = \lambda m - \nu_t(H)$$

Indeed, the hypergraph H^* is h-uniform, and from Propositions 1, 2 and 3,

$$\rho_s(H) = \tau_s(H^*) = \lambda m - \bar{\alpha}_t(H^*) = \lambda m - \nu_t(H).$$

By duality we obtain:

Theorem 1$'$. *Every hypergraph H satisfies:*

$$\bar{\alpha}(H) = \min_{k \geq 1} \frac{\bar{\alpha}_k(H)}{k} \leq \max_{A \subseteq X} \frac{|A|}{r(H_A)} \leq \max_{k \geq 1} \frac{\bar{\alpha}_k(H)}{k} = \alpha^*(H)$$

$$= \min_{k} \frac{\rho_k(H)}{k} \leq \min_{H' \subseteq H} \frac{m(H')}{\delta(H')} \leq \max_{k \geq 1} \frac{\rho_k(H)}{k} = \rho(H).$$

Corollary. *The edges of a hypergraph H with the dual König property are coverable with k edges if and only if*

$$kr(H_A) \geq |A| \quad (A \subset X)$$

Indeed, if $\overline{\alpha}(H) = \rho(H)$ *we have*

$$\rho(H) = \max_{A \subseteq X} \frac{A}{r(H_A)} \leq k.$$

This is equivalent to the stated condition.

Example: Consider the celebrated problem of Gauss: what is the maximum number of queens which may be placed on an 8×8 chessboard such that no two lie in the same row, column or diagonal. If we consider diagram A we see that it is possible to place 8 queens in such a manner, and 8 is clearly the maximum. In other words, the hypergraph H on the set of squares, whose edges are the rows, columns and diagonals of the chessboard, satisfies $\overline{\alpha}(H) = 8$. Clearly $\rho(H) = 8$, since the 8 columns constitute a covering, and the hypergraph H has the dual König property: $\overline{\alpha}(H) = \rho(H)$.

More difficult is the following problem: what is the minimum number of queens necessary to cover every row, column and diagonal at least once? Clearly $\nu(H) = 14$, since we may form a matching with the 7 white diagonals parallel to the leading white diagonal and the 7 black diagonals parallel to the leading black diagonal. Further, we also have $\tau(H) = 14$, a transversal of 14 elements being represented in diagram B. Hence $\nu(H) = \tau(H)$, and the hypergraph H satisfies the König property.

Note that this is not the same as the domination problem: what is the minimum number of queens necessary to dominate all the squares? The answer is 5, and the solution of diagram C corresponds to a maximal strongly stable set of minimum weight: thus $\overline{\alpha}'(H) = 5$.

We may also ask the question: is it possible to place 16 queens in such a way that each row, column and diagonal contains at most two queens? Since $\overline{\alpha}_2(H) = 2\overline{\alpha}(H) = 16$ this is clearly possible if we allow two queens to occupy the same square. However, diagram D gives as a solution a 0-1 vector, that is to say an optimal strongly 2-stable "set".

Finally, we may consider the problem: does there exists a 2-transversal which is a set of 28 queens, all placed on different squares? An optimal 2-transversal with 0-1 coordinates is represented in diagram E.

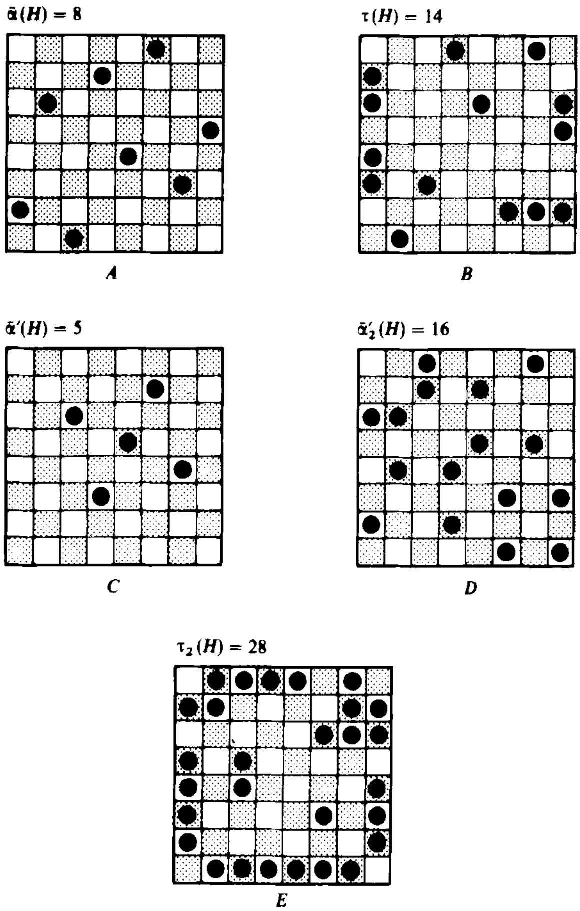


Figure 3

2. Fractional matchings of a graph

We now suppose that the hypergraph is a simple graph denoted by $G = (X,E)$. From Theorem 1, we have

$$\nu(G) = \min_{k \geq 1} \frac{\nu_k(G)}{k} \leq \max_{G'} \frac{m(G')}{\Delta(G')} = \tau^*(G) =$$

$$= \min_{k \geq 1} \frac{\tau_k(G)}{k} \leq \max_{k \geq 1} \frac{\tau_k(G)}{k} = \tau(G)$$

Theorem 2. *Every graph G satisfies*

$$\tau^*(G) = \frac{\nu_2(G)}{2} = \frac{\tau_2(G)}{2}.$$

Further, there exists a maximum 2-matching, $H \subset 2G$ whose connected components are isolated vertices, pairs of parallel edges, and odd cycles.

For such a 2-matching H, there exists a minimum fractional transversal t such that $t(x) = 0$ if x is an isolated vertex of H; $t(x) = 0$, $t(y) = 1$ (or $t(x) = t(y) = \frac{1}{2}$) if x and y are the endpoints of a pair of parallel edges of H; $t(x) = \frac{1}{2}$ if x belongs to an odd cycle of H.

Proof: Let $H \subset 2G$ be a 2-matching with $m(H)$ maximum. Each connected component of H which is a path of even length or an even cycle may be replaced by pairs of parallel edges without changing $m(H)$. No component of H is a path of odd length, since we could then augment $m(H)$ by replacing it by pairs of parallel edges. We may thus suppose that H is of the indicated type.

We now label each vertex of G with a 0, a 1 or a $\frac{1}{2}$, step by step, according to the following rules:

(1) an isolated vertex of H is labelled 0;

(2) a vertex which is adjacent *in* G to a vertex labelled 0 is labelled 1;

(3) a vertex which is adjacent *in* H to a vertex labelled 1 is labelled 0;

(4) each vertex which cannot be labelled by rules 1, 2, 3 is labelled $\frac{1}{2}$.

Observe that an odd path starting at an isolated vertex of H followed alternately by edges of $G-H$ and double edges of H cannot terminate in an isolated vertex of H: otherwise, by replacing in H the double edges of this path by the path itself we obtain a 2-matching H' with $m(H') = m(H)+1$, contradicting the maximality of H. Similarly, an odd path of this type cannot terminate in an odd cycle of H. Finally an odd path of this type cannot contain any other vertex labelled 0.

Hence a single label $t(x)$ may be given to each vertex x and t indeed takes the values given in the statement. From rule 2, the function $t(x)$ is a fractional transversal of G, and, by Theorem 1, we obtain:

$$\frac{m(H)}{2} = \frac{\nu_2(G)}{2} \le \tau^*(G) \le \sum_{x \in X} t(x) = \frac{m(H)}{2}$$

Thus we have equality throughout, which shows that $t(x)$ is a maximum fractional transversal of G, and that

$$\tau^*(G) = \frac{\nu_2(G)}{2} = \frac{\tau_2(G)}{2.}$$

Theorem 3 (Lovász [1975]). *Every graph G satisfies*

$$\tau^*(G) \le \frac{1}{2}(\nu(G)+\tau(G)).$$

Proof: Let T be a minimum transversal of the graph $G = (X,E)$; the set $S = X-T$ is then a maximum stable set. Let k be the maximum number of disjoint edges having an end in S. From König's Theorem on maximum matchings in bipartite graphs (cf. *Graphs*, chapter 7), there exists a subset A_0 of S such that

$$|S-A_0|+|\Gamma_G A_0| = \min_{A \subseteq S} (|S-A|+|\Gamma_G A|) = k.$$

Put

$$t(x) = \begin{cases} 0 \text{ if } x \in A_0 \\ 1 \text{ if } x \in \Gamma_G A_0 \\ \frac{1}{2} \text{ if } x \in X-(A_0 \cup \Gamma_G A_0) \end{cases}$$

Clearly, $t(x)$ is a fractional transversal of G; whence:

$$2\tau^*(G) \le 2\sum_{x \in X} t(x) = |T|+|\Gamma_G A_0|+|S-A_0|$$

$$= \tau(G)+k \le \tau(G) + \nu(G)$$

from which we deduce the result.

Corollary. *For a graph G the following conditions are equivalent:*

(1) $\tau^*(G) = \tau(G)$

(2) $\nu(G) = \tau(G)$

Proof: Since, from the fundamental inequalities, (2) implies (1), it suffices to show that (1) implies (2). Let G be a graph satisfying (1); from Theorem 3,

$$\tau(G) = \tau^*(G) \leq \frac{1}{2}(\nu(G)+\tau(G)) \leq \frac{1}{2}(\tau(G)+\tau(G)).$$

We thus have equality throughout, which implies (2).

Remark: It is not true that $\tau^*(G) = \nu(G)$ implies $\nu(G) = \tau(G)$. For example, $\tau^*(K_4) = \nu(K_4) = 2$, but $\tau(K_4) = 3$.

An optimal 2-matching H of the form given in Theorem 2 determines an optimal fractional matching q; the set of edges e of G with $q(e) \neq 0$ defines a partial subgraph of G whose connected components are: isolated vertices, isolated edges and odd cycles. Such an optimal fractional matching is said to be *canonical*. Balinski [1970] showed that the canonical matching are the extreme points of the matching polytope. We have:

Theorem 4 (Uhry [1975]). *Let $G = (X,E)$ be a graph, and let q be a canonical fractional matching such that the set of edges e with $q(e) = \frac{1}{2}$ is minimal with respect to inclusion. Then we obtain a maximum matching M of G on taking the union of $M_0 = \{e/q(e) = 1\}$ and all the M_i's where M_i denotes a maximum matching of the odd cycle μ_i of $\{e/e \in E;\ q(E) \neq 0\}$.*

(*) **Proof:** Let $\mu_1, \mu_2, \ldots$ be the odd cycles formed by those edges e with $q(e) = \frac{1}{2}$; denote by X_i the set of vertices of μ_i and set $X_0 = X - \bigcup_{i \geq 1} X_i$.

Clearly M_0 is a maximum matching of the subgraph G_{X_0}. We shall show that $M = M_0 \cup M_1 \cup M_2 \cup \cdots$ is a maximum matching of G.

Suppose that the matching M is not maximum. From the alternating path lemma (cf. *Graphs*, chapter 7 §1), there exists an alternating path $\mu[a,b]$ between two vertices a and b unsaturated by M. In the subgraph of G induced by X_0, the edges of M form a maximum matching, and consequently the chain $\mu[a,b]$ meets at least one of the X_i's, say X_1. Further, since the subgraph of G induced by X_1 contains a single unsaturated vertex by M, one of the ends of $\mu[a,b]$ is in $X-X_1$, say b. Let a' be the last point of the path $\mu[a,b]$ which is in X_1.

Since no edge of M joins X_1 and $X-X_1$, we may suppose perhaps after modifying the maximum matching M_1 of X_1, that a' is unsaturated: in other words we may suppose that $a' = a$.

Case 1: $a \in X_1$, $b \in X_2$.

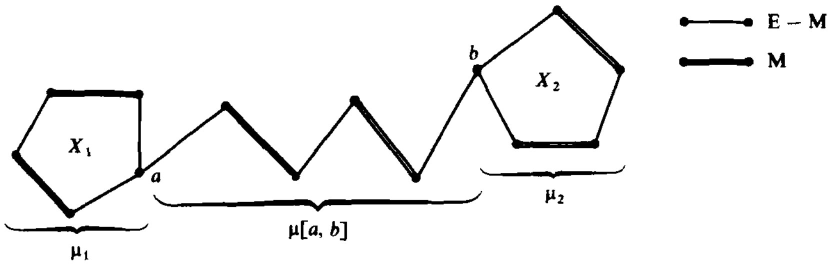


Figure 4

For every set $F \subset E$, denote the characteristic function of F by $\phi_F(e)$ and let

$$q'(e) = \begin{cases} 1-q(e) & \text{if } e \in \mu[a,b] \\ \phi_{M_1}(e) & \text{if } e \in \mu_1 \\ \phi_{M_2}(e) & \text{if } e \in \mu_2 \\ q(e) & \text{otherwise} \end{cases}$$

Clearly $q'(e)$ is a fractional matching of H. Since

$$\Sigma q'(e) = \Sigma q(e),$$

q' is also an optimal fractional matching. As q' has fewer edges weighted $\frac{1}{2}$ than q, this contradicts the definition of q.

Case 2: $a \in X_1$, $b \in X_0$.

Let

$$q'(e) = \begin{cases} 1-q(e) & \text{if } e \in \mu[a,b] \\ \phi_{M_1}(e) & \text{if } e \in \mu_1 \\ q(e) & \text{otherwise} \end{cases}$$

Clearly q' is a fractional matching H, and

$$\Sigma q'(e) - \Sigma q(e) = 1 - \frac{1}{2} > 0.$$

This contradicts the optimality of q.

In each case we obtain a contradiction, which shows that the matching M is maximum, as required.

The following theorem may be used to characterise those graphs G for which $\nu(G) = \tau^*(G)$. Let M be a maximum matching of the graph $G = (X,E)$. A cycle μ of G is said to be *isolated by* M is no edge of M joins μ and $X-\mu$. Let $s(M)$ be the maximum number of pairwise disjoint odd cycles of G isolated by M.

Theorem 5 (Balas [1981]). *Every graph G satisfies:*

$$\tau^*(G) = \nu(G) + \frac{1}{2} \max_M s(M)$$

Proof: Let q be a canonical fractional matching having a minimal set of edges e with $q(e) = \frac{1}{2}$; let $\mu_1, \mu_2, \ldots, \mu_s$ be the odd cycles generated by these edges. The matching M obtained from q as in the statement of Theorem 4 satisfies

$$\tau^*(G) - \nu(G) = \sum_e q(e) - |M| = \frac{s}{2} \leq \frac{1}{2} \max_M s(M)$$

(since M isolates the cycles $\mu_1, \mu_2, \ldots, \mu_s$).

Further, suppose there exists a matching M' with $|M| = |M'|$ and $s(M') > s$; then M' may be obtained from a canonical fractional matching q', and

$$\sum_e q'(e) = |M'| + \frac{1}{2} s(M') > |M| + \frac{s}{2} = \Sigma q(e).$$

This contradicts the optimality of q. Thus $s = \max s(M)$ and the stated equality follows.

To illustrate this result, consider the graph of Figure 5. It has a maximum matching M_1 which does not isolate the pentagon, but also a maximum matching M_2 which does. Thus, $\max s(M) = 1$, and we may thus find a fractional matching $q(e)$ of value $\nu(G) + \frac{1}{2} = \frac{7}{2}$ (see Figure 5).

Corollary 1. *A graph G satisfies* $\nu(G) = \tau^*(G)$ *if and only if no maximum matching isolates an odd cycle.*

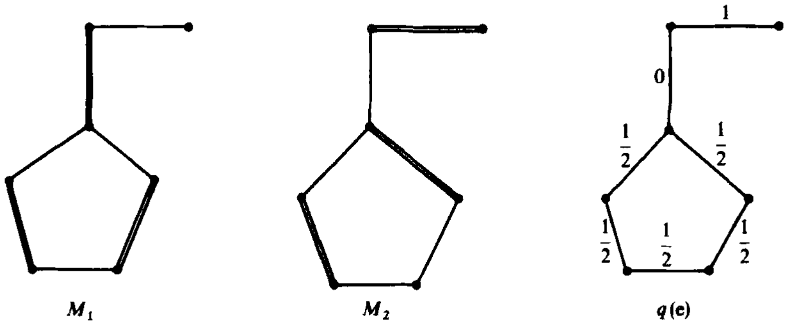


Figure 5

Indeed, in this case, $\max\limits_{M} s(M) = 0$.

Corollary 2 (Lovász [1975]). *Every graph G satisfies*

$$\text{(1)} \quad \tau^*(G) \leq \frac{3}{2}\,\nu(G)$$

Equality holds in (1) *if and only if G is the union of pairwise disjoint triangles.*

Proof: It is clear that if G consists of p vertex-disjoint triangles, then $\tau^*(G) = \frac{3p}{2}$ and $\nu(G) = p$, giving equality in (1).

If G is not of this type, let M be a maximum matching of G which maximizes $s(M)$. Each of the $s(M)$ odd cycles isolated by M contains at least one edge of M, so

$$\tau^*(G) = \nu(G) + \frac{1}{2}s(M) \leq \nu(G) + \frac{1}{2}\nu(G) = \frac{3}{2}\nu(G)$$

Equality in (1) implies that each odd cycle is a triangle and contains exactly one edge of M. These triangles are disjoint since any extra edge would create an alternating path between two unsaturated vertices in distinct triangles, contradicting the maximality of M.

We will now prove a result which gives a characterisation of graphs G with $\tau^*(G) = \tau(G)$.

Let M be a maximum matching of G; an odd cycle of length $2k+1$ containing k edges of M is called a *lentil*; its *base* is the vertex which is not adjacent to any of these k edges.

A *monocle* is the disjoint sum $\mu_1 + \mu_2$ of a lentil μ_1 and an alternating path μ_2 of even length joining a vertex unsaturated by M to the base of the lentil μ_1 (cf. Figure 6).

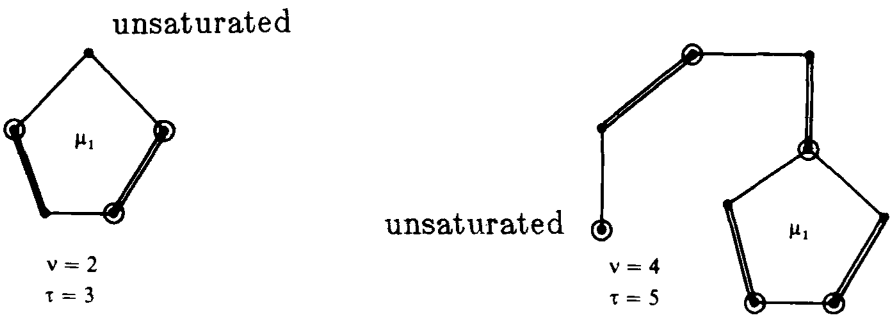


Figure 6. Monocles

If two (not necessarily disjoint) lentils μ_1 and μ_2 are joined at the bases by an odd alternating path μ_3, their sum $\mu_1 + \mu_2 + \mu_3$ is called a *binocle* (cf. Figure 7).

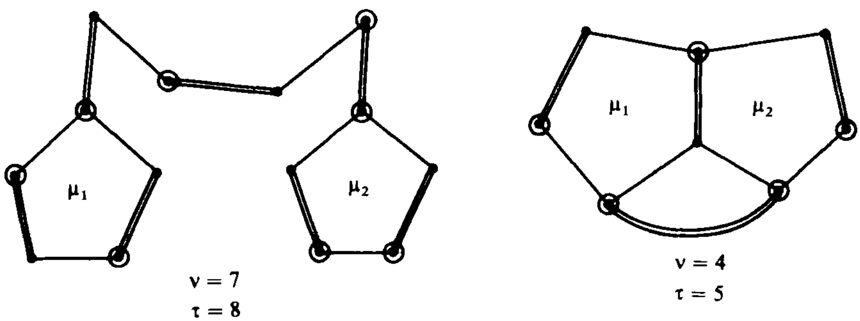


Figure 7. Binocles

Recall that an alternating path (relative to M) is a sequence of distinct edges alternately from M (the "thick" edges) and from $E-M$ (the "thin" edges).

For every maximum matching M, we say that a vertex is *thin* if may be reached by an odd alternating path from an unsaturated vertex (and not by an even path). We say it is *thick* if it may be reached by an even alternating path from an unsaturated vertex (and not by an odd path). We say that it is *mixed* if it may be reached by an even alternating path and by an odd alternating path. We say that it is *inaccessible* if it cannot be reached by an alternating path from an unsaturated vertex. Thus an unsaturated vertex is thick or mixed; if there are no unsaturated points then all the vertices of the graph are inaccessible.

The following lemmas are, in fact, in a weaker form, general properties of matchings (Gallai [1950], Berge [1967]).

Lemma 1: *Let G be a graph without inaccessible points with respect to a maximum matching M. Then there is a mixed point if and only if G has a monocle.*

Indeed, the first mixed point reached by an alternating path starting at an unsaturated point is always the base of a monocle.

Lemma 2: *If G contains nothing but thick or thin points relative to a maximum matching M, the set T of the thin vertices constitutes a minimum transversal; further $|T| = \nu(G)$.*

Indeed, each vertex adjacent to a thick vertex is thin, thus the set T is a transversal; each edge of the matching contains a thick vertex and a thin vertex, and the unsaturated vertices are all thick. Thus $|T| = |M| = \nu(G)$.

Lemma 3: *Let C be a connected component of the subgraph of G generated by the inaccessible points relative to a matching M; then no edge of M joins C to $X-C$, and each vertex of $X-C$ adjacent to C is a thin vertex.*

(Clear).

Theorem 6 (Sterboul [1978]; Deming [1979]). *For a graph G, the following conditions are equivalent:*

(1) $\nu(G) = \tau(G)$;

(2) *For every maximum matching M, the graph G has no monocle or binocle;*

(3) *There exists a maximum matching M for which G has no monocle or binocle.*

(*) Proof:

(1) implies (2). Suppose that $\nu(G) = \tau(G)$. Let M be a maximum matching for which the graph G has a monocle $\mu_1 + \mu_0$ where μ_1 is a lentil with base a and $\mu_0 = \mu[a,b]$ the alternating path joining point a to an unsaturated point b. In the matching $M-(M \cap \mu_0) + (\mu_0 - M)$ which is also maximum, the odd cycle μ_1 is isolated, hence $\max s(M) \geq 1$. Thus, from Theorem 5, $\nu(G) \neq \tau^*(G)$, which contradicts $\nu(G) = \tau(G)$.

Now let M be a maximum matching for which the graph G has a binocle $\mu_1 + \mu_2 + \mu[a,b]$ where $\mu[a,b]$ is the alternating path joining the two bases of the lentiles μ_1 and μ_2.

- If a vertex of the binocle is joined by an alternating path to an unsaturated vertex z, we may obtain, by interchanging the thick edges and the thin edges along an alternating path starting at z, a maximum matching which isolates one of the lentils, which contradicts $\nu(G) = \tau(G)$.

- Otherwise, let T be a minimum transversal of G, and let x be a vertex of $\mu[a,b]$ which belongs to T. In the graph G' obtained from G by adjoining a vertex x_0 and the edge $[x_0,x]$, the matching M is still maximum (since no alternating path joins x_0 to another isolated vertex), and T is still a minimum transversal. Thus $\nu(G') = |M| = |T| = \tau(G')$. By interchanging the thick edges with the thin edges along an alternating path $[x_0,x] + \mu[x,b]$ we create an odd cycle μ_2 isolated by a maximum matching M'; thus $s(M') \geq 1$ and $\nu(G') \neq \tau^*(G')$: contradiction.

(2) implies (3). Obvious.

(3) implies (1). Indeed, let G be a connected graph with $\nu(G) \neq \tau(G)$ and let M be a maximum matching for which G contains no monocle or binocle; suppose that G is of minimum order with these conditions: we now deduce a contradiction

Case 1: G has an unsaturated point relative to M.

If the set of inaccessible points is $A \subset X$, we have $|A| \neq |X|$. From lemma 1, G contains no mixed points, and hence the subgraph $\overline{G}$ induced by $X-A$ has only thick or thin vertices. The matching $\overline{M}$ given by the restriction of M to $\overline{G}$ is a maximum matching of $\overline{G}$ (since no alternating path joins two distinct unsaturated points). From lemma 2, the set $\overline{T}$ of thin vertices of $\overline{G}$ is a transversal with $|\overline{T}| = |\overline{M}|$; moreover $\overline{T}$ meets each edge joining A and $X-A$.

The subgraph $\overline{\overline{G}} = G_A$ admits as a maximum matching the restriction $\overline{\overline{M}}$ of M (since $\overline{\overline{G}}$ contains no unsaturated vertices) and contains no monocle or binocle; thus, by the induction hypothesis it has a transversal $\overline{\overline{T}}$ with $|\overline{\overline{T}}| = |\overline{\overline{M}}|$. The set $\overline{T} \cup \overline{\overline{T}}$ is a transversal of G, and $|\overline{T} \cup \overline{\overline{T}}| = |\overline{M} \cup \overline{\overline{M}}| = |M|$, contradicting the assumption that $\nu(G) \neq \tau(G)$.

Case 2: G has no unsaturated vertices relative to M. Let G' be the graph formed by adjoining to G a vertex x_0 and an edge $[x_0,x_1]$ joining x_0 and a vertex x_1 in a minimum transversal T of G. Since G' has only one unsaturated point, we know, from the alternating path lemma, that M is also a maximum matching of G'. Further, T is also a minimum transversal of G'. Thus $\nu(G') = |M| < |T| = \tau(G')$.

The graph G' has mixed points (since otherwise we would see as in case 1 that $\nu(G') = \tau(G')$, a contradiction). From lemma 1 we deduce that G contains a monocle. Let μ_1 be its lentil, and b_1 its base. We have $b_1 \neq x_0$ (since b_1 is of degree ≥ 2).

Let G'' be the graph obtained from the original graph G by adjoining a vertex y_0 and the edge $[y_0,b_1]$. If G'' contains no mixed points we see as above that $\nu(G'') = \tau(G'')$ which implies $\nu(G) = \tau(G)$: contradiction.

If G'' contains a mixed point, the first mixed point along an alternating path from y_0 is the base of a lentil μ_2; clearly μ_2 forms a binocle of G with μ_1, which gives a contradiction.

3. Fractional transversal number of a regularisable hypergraph

Let $H = (E_1,E_2,\ldots,E_m)$ be a hypergraph on X. For an integer $k \geq 0$, *multiplying the edge E_i by k* consists of replacing the edge E_i in H by k identical copies of E_i. If $k = 0$, this operation becomes deletion of the edge E_i.

A hypergraph H is *regular* if all the vertices have the same degree; H is *regularisable* if a regular hypergraph may be obtained from H by multiplying each edge E_i by an integer $k_i \geq 1$. Finally, H is *quasi-regularisable* if a regular hypergraph may be obtained by multiplying each edge E_i by an integer $k_i \geq 0$; note that this regular hypergraph H' cannot contain a vertex of degree 0, since this is incompatible with the definition of "hypergraph".

Some examples of graphs with these properties are given in Figure 8.

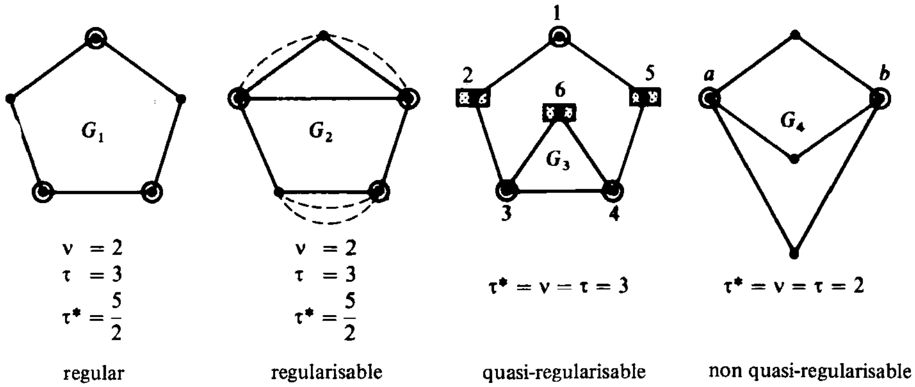


Figure 8

Clearly we have: regular ⇒ regularisable ⇒ quasi-regularisable.

Theorem 7: *For an r-uniform hypergraph $H = (E_1, E_2, \ldots, E_m)$ on X, $|X| = n$, the following properties are equivalent:*

(1) *H is quasi-regularisable;*

(2) $\tau^*(H) = \frac{n}{r}$.

Proof.

(1) implies (2). If the hypergraph H is quasi-regularisable, there exists a regular s-matching $H' \subset sH$; by counting the edges of the incidence graph of the edges of H' in two different ways, we obtain $ns = rm(H')$. Thus

$$\frac{n}{r} = \frac{m(H')}{s} \leq \tau^*(H) \leq \frac{n}{r}$$

(since $t(x) \equiv \frac{1}{r}$ is a fractional transversal of H).

Thus we have equality throughout, and consequently

$$\tau^*(H) = \frac{n}{r}$$

(2) implies (1). Let s be the integer ≥ 1 such that

$$\frac{\nu_s(H)}{s} = \max_{k \geq 1} \frac{\nu_k(H)}{k}$$

Let $H' \subset sH$ be an s-matching such that $m(H') = \nu_s(H)$. From (2),

$$\frac{m(H')}{s} = \frac{\nu_s(H)}{s} = \tau^*(H) = \frac{n}{r}$$

Thus $rm(H') = n\Delta(H')$, which shows that the hypergraph H' is regular, thus H' is quasi-regularisable.

Remark: Hence, in Figure 8, G_3 is quasi-regularisable because the matching [1,2], [3,6], [4,5] is perfect; the graph G_4 is not, since the function $t(x) = 1$ for $x \in \{a,b\}$ and $t(x) = 0$ for $x \in X - \{a,b\}$ is a fractional transversal with value $2 < \frac{n}{2} = \frac{5}{2}$.

When the hypergraph is a graph we can refine Theorem 7 as follows:

Theorem 8. *For a graph G of order n, the following conditions are equivalent:*

(1) *G is quasi-regularisable;*

(2) $\tau^*(G) = \frac{n}{2}$;

(3) *G admits a partial graph H whose components consist of 2-cliques and odd cycles;*

(4) $|\Gamma_G S| \geq |S|$ *for every stable set S of G.*

Proof.

(1) implies (2).

If the graph G of order n satisfies (1), then there exists a regular multigraph $H \subset kG$ of degree k. By counting the edges of the incidence graph of H in two different ways, we obtain:

$$kn = 2m(H).$$

Thus

$$\frac{n}{2} = \frac{m(H)}{k} \leq \frac{\nu_k(G)}{k} \leq \tau^*(G) \leq \frac{\tau_2(G)}{2} \leq \frac{n}{2}$$

since $t(x) \equiv 1$ is always a 2-transversal of G.

Thus we have equality throughout, and $t(x) \equiv 1$ is an optimal 2-transversal.

(2) implies (3).

Let G be a graph satisfying (2). Then, from Theorem 2

$$\frac{\nu_2(G)}{2} = \tau^*(G) = \frac{n}{2}$$

Thus $\nu_2(G) = n$, whence (3) holds.

(3) implies (4).

Indeed, for every stable set S of G,

$$|\Gamma_G S| \geq |\Gamma_H S| \geq |S|.$$

(4) implies (1).

Indeed, let G be a graph satisfying (4); let $t(x)$ be a 2-transversal of G. The set $S = \{x/t(x)=0\}$ is stable, and $\Gamma_G S \subset \{x/t(x)=2\}$. Thus

$$\sum_x t(x) = n + \sum_x (t(x)-1) \geq n + |\Gamma_G S| - |S| \geq n.$$

Thus the 2-transversal $t'(x) \equiv 1$ is optimal, whence, from Theorem 2,

$$\frac{\nu_2(G)}{2} = \frac{\tau_2(G)}{2} = \frac{n}{2}$$

Thus $\nu_2(G) = n$, which shows that G is quasi-regularisble.

Theorem 9. (Fulkerson-McAndrew-Hoffman Theorem). *Let G be a connected graph of even order such that every pair of disjoint odd cycles are joined by an edge. Then a necessary and sufficient condition for G to have a perfect matching is that every stable set S satisfy $|\Gamma_G S| \geq |S|$.*

Proof: The condition is clearly necessary. It is also sufficient since this is condition (4) of Theorem 8, which implies that G admits a partial graph whose components are just isolated edges and odd cycles. The cycle components may be grouped in pairs (since n is even) and each group of two odd cycles joined by an edge is replaceable by a perfect matching. We thus have the result.

For regularisable bipartite graphs we may easily find analogous conditions to those of Theorem 8. The following characterises regularisable graphs by the uniqueness of the optimal 2-transversal. Other characterisations exist, notably due to Pulleyblank [1980], [1981].

Theorem 10 (Berge [1978]). *For a connected graph G of order n, the following conditions are equivalent:*

(1) *G is regularisable and not bipartite;*

(2) *$\tau^*(G) = \frac{n}{2}$ and $t(x) \equiv 1$ is the unique optimal 2-transversal;*

(3) *$|\Gamma_G S| > |S|$ for every stable set S of G;*

(4) *$|\Gamma_G A| > |A|$ for every set $A \subset X$, $A \neq \emptyset$, $A \neq X$.*

Proof.

(1) implies (2). If G satisfies (1), there exists a regular multigraph H obtained from G by multiplication of edges; the 2-transversal $t(x) \equiv 1$ is optimal for G from condition (2) of Theorem 8 (since regularisability implies quasi-regularisability).

Suppose that there exists another optimal 2-transversal $t'(x)$, that satisfy $t'(X) = n$; we deduce a contradiction. The set $A_0 = \{x / t'(x)=0\}$ is stable, and has the same cardinality as $A_2 = \{x / t'(x)=2\}$ (since $t'(X) = n$). Further, $\Gamma_G A_0 \subset A_2$. Since H is regular, we have

$$\Delta(H)|A_0| = \sum_{x \in A_0} m_H(x,A_2) = \sum_{x \in A_2} m_H(x,A_0) \leq \Delta(H)|A_2| = \Delta(H)|A_0|$$

We thus have equality throughout, so every edge with one end in A_2 has its other end in A_0; the subgraph $G_{A_0 \cup A_2}$ is thus equal to G (since G is connected) and this is a bipartite graph with 2 classes of the same cardinality: contradiction.

(2) implies (3).

Let S be a stable set in G; there exists a multigraph $H \subset 2G$ corresponding to a canonical 2-matching of the form indicated in Theorem 2. Since $\nu_2(G) = n$, no components of G is an isolated point. Thus

$$|\Gamma_G S| \geq |\Gamma_H S| \geq |S|.$$

We cannot have $|\Gamma_G S| = |S|$ since this would imply the existence of another transversal t' defined by

$$t'(x) = \begin{cases} 0 & \text{if } x \in S \\ 2 & \text{if } x \in \Gamma_G S \\ 1 & \text{if } x \notin S \cup \Gamma_G S; \end{cases}$$

t' is also optimal (since $t'(X) = n$), and this contradicts the uniqueness of the optimal 2-transversal. Thus

$$|\Gamma_G S| > |S|.$$

(3) implies (4).

Let A be a set of vertices, $A \neq \emptyset, X$. Let S be the set of isolated vertices in the subgraph G_A. If $S = \emptyset$, we have $m_G(A, X-A) \neq 0$ (since G is connected); thus $\Gamma_G A$ contains A and at least one point of $X-A$; thus $|\Gamma_G A| > |A|$.

If $S \neq \emptyset$, we have $|\Gamma_G S| > |S|$ from (3), so

$$|\Gamma_G A| \geq |\Gamma_G S| + |A-S| > |S| + |A-S| = |A|.$$

(4) implies (1).

Let H be the bipartite graph obtained by taking two copies X and $\overline{X}$ of the set of vertices of G, and joining $x \in X$ to $\bar{y} \in \overline{X}$ if and only if $[x,y]$ is an edge of G. Every set $A \subset X$ with $A \neq \emptyset, X$ satisfies $|\Gamma_H A| = |\Gamma_G A| > |A|$.

It suffices to show that an edge $[a,\bar{b}]$ of H appears in at least one perfect matching of H (since such a matching defines a 2-matching G_{ab} of G containing the edge $[a,b]$, and $\sum_{ab} G_{ab}$ is a regular multigraph, which shows that G is regularisable).

Indeed, in the subgraph H' of H induced by $X \cup \overline{X} - \{a,\bar{b}\}$, every set $A \subset X - \{a\}$ satisfies

$$|\Gamma_{H'} A| = |\Gamma_H A_- \{\bar{b}\}| \geq |\Gamma_H A| - 1 \geq |A|.$$

Hence H' has a perfect matching (from König's theorem), so H has a perfect matching which contains the edge $[a,b]$.

Theorem 11 (Jaeger, Payan [1978]). *Let G be a connected graph not containing a $K_{1,3}$ as an induced subgraph. Then G is regularisable if and only if it has no "hanging" vertex, (that is to say a vertex of degree 1) and is not isomorphic to the graph G_1 consisting of an even cycle of the form* $[0,1,2,\ldots,2p-1,0]$ *with a non-empty set of chords of the form* $[2i,2i+2]$.

Proof: Observe first that the graph G_1 above is $K_{1,3}$-free. Since the set $S = \{1,3,5,...\}$ satisfies $|\Gamma_{G_1}S| = |S|$ and as G_1 is non-bipartite, it is clear that G_1 is non-regularisable from condition (3) of Theorem 10.

Let G be a connected graph without $K_{1,3}$ which is not isomorphic to G_1. Suppose further that G has a stable set with $|\Gamma_G S| \leq |S|$. If $x \in S$ the number of edges between x and $\Gamma_G S$ is $m_G(x,\Gamma_G S) \geq 2$ (since G has no hanging vertices); if $y \in \Gamma_G S$ we have $m_G(y,S) \leq 2$ (since G has no $K_{1,3}$). Thus

$$2|\Gamma_G S| \leq 2|S| \leq \sum_{x \in S} m_G(x,\Gamma_G S) = \sum_{y \in \Gamma_G S} m_G(y,S) \leq 2|\Gamma_G S|.$$

We thus have equality throughout, and consequently

$$|S| = |\Gamma S|.$$

The equalities show further that for every $x \in S$ and every $y \in \Gamma_G S$, $m(x,\Gamma_G S) = m_G(y,S) = 2$; thus the edges of G between S and $\Gamma_G S$ form an even cycle. The only possible additional edges join two vertices of $\Gamma_G S$ and are triangular chords of the cycle (otherwise G contains a $K_{1,3}$).

Hence G is isomorphic to G_1, which contradicts the hypothesis.

Thus we have shown that $|\Gamma_G S| > |S|$ and, from Theorem 10, the graph G is regularisable and non-bipartite.

4. Greedy transversal number

Let H be a simple hypergraph; for a vertex x we denote by $H(x)$ the set of edges of H which contain x. To obtain a transversal of small cardinality, we may use the *greedy algorithm*:

1. choose a vertex x_1 of maximum degree in $H_1 = H$;
2. choose a vertex x_2 of maximum degree in $H_2 = H_1 - H_1(x_1)$;
3. choose a vertex x_3 of maximum degree in $H_3 = H_2 - H_2(x_2)$;
4. etc.

We stop when the hypergraph H_{k+1} has all its vertices of degree 0; the set $T = \{x_1,x_2,...,x_k\}$ is then a transversal of H. The maximum cardinality of a transversal obtained by a greedy algorithm is called the *greedy transversal number*, and is denoted by $\tilde{\tau}(H)$.

The following theorem, in a slightly improved form, is a result found independently by Stein [1974] and by Lovasz [1975].

Theorem 12. *For a hypergraph H of maximum degree Δ,*

$$\tau(H) \le \tilde{\tau}(H) \le (1+\frac{1}{2}+\cdots+\frac{1}{\Delta}) \max_{H'\subseteq H} \frac{m(H')}{\Delta(H')} \le (1+\log\Delta)\tau^*(H)$$

Proof: Let T be a transversal of H with $|T| = \tilde{\tau}(H)$ which has been obtained by the greedy algorithm; let t_λ be the number of steps taken to choose a vertex of degree λ. If H has maximum degree Δ, we have

$$\tilde{\tau}(H) = |T| = t_1+t_2+\cdots+t_{\lambda+1}+\cdots+t_\Delta.$$

For $\lambda < \Delta$, put $t_\Delta+t_{\Delta-1}+\cdots+t_{\lambda+1} = k$. The $(k+1)$-th step consists of finding a vertex x_{k+1} of maximum degree in the partial hypergraph H_{k+1} and we observe that $\Delta(H_{k+1}) \le \lambda$. By counting the number of remaining edges that all the following steps will remove, we obtain:

$$\lambda t_\lambda + (\lambda-1)t_{\lambda-1}+\cdots+2t_2+t_1 = m(H_{k+1}) \le \lambda \frac{m(H_{k+1})}{\Delta(H_{k+1})} \le \lambda \max_{H'\subseteq H} \frac{m(H')}{\Delta(H')}$$

We may rewrite this as:

$$(\frac{1}{\lambda} - \frac{1}{\lambda+1})(t_1+2t_2+\cdots+\lambda t_\lambda) \le \frac{1}{\lambda+1} \max \frac{m(H')}{\Delta(H')}$$

These inequalities are satisfied for $\lambda = 1,2,\ldots,\Delta-1$ and we obtain a system of inequalities:

$$(1-\frac{1}{2})t_1 \le \frac{1}{2} \max \frac{m(H')}{\Delta(H')}$$

$$(\frac{1}{2}-\frac{1}{3})(t_1+2t_2) \le \frac{1}{3} \max \frac{m(H')}{\Delta(H')}$$

$$(\frac{1}{3}-\frac{1}{4})(t_1+2t_2+3t_3) \le \frac{1}{4} \max \frac{m(H')}{\Delta(H')}$$

....

$$(\frac{1}{\Delta-1}-\frac{1}{\Delta})(t_1+2t_2+\cdots+(\Delta-1)t_{\Delta-1}) \le \frac{1}{\Delta} \max \frac{m(H')}{\Delta(H')}$$

$$\frac{1}{\Delta}(t_1+t_2+\cdots+\Delta t_\Delta) = \frac{m(H)}{\Delta(H)} \le \max \frac{m(H')}{\Delta(H')}$$

Summing the respective sides of these inequalities, we obtain on the left $\sum_{\lambda=1}^{\Delta} t_\lambda$, and on the right,

$$(1+\frac{1}{2}+\cdots+\frac{1}{\Delta}) \max \frac{m(H')}{\Delta(H')} \leq (1+\log\Delta)\tau^*(H),$$

whence, finally,

$$\tilde{\tau}(H) = \sum_{\lambda=1}^{\Delta} t_\lambda \leq (1+\log\Delta)\, \tau^*(H).$$

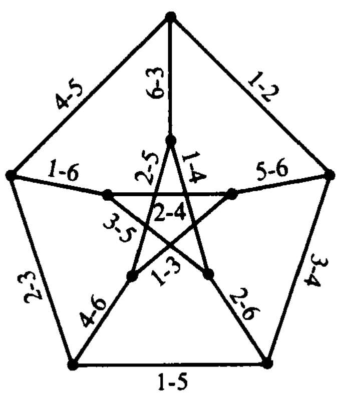
Application: Fractional chromatic index of a graph.

Consider a multigraph G without loops. The chromatic index $q(G)$ is the least number of colours necessary to colour the edges of G such that two edges of the same colour are never adjacent. The *fractional chromatic index* is defined to be

$$q^*(G) = \min_{k\geq 1} \frac{q(kG)}{k}$$

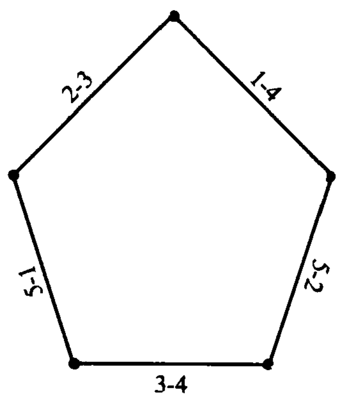
Clearly $q^*(G) \geq \Delta(G)$.

For the Petersen graph P_{10} we see that $q(2P_{10}) = 6$ (cf. Figure 9), so $q^*(P_{10}) = \dfrac{q(2P_{10})}{2} = 3 = \Delta(P_{10})$. For the odd cycle C_5 we have $q(2C_5) = 5$, so $q^*(C_5) = \dfrac{5}{2}$ (cf. Figure 10); more generally, $q^*(C_{2p+1}) = 2 + \dfrac{1}{p} > \Delta(G)$.



P_{10}

Figure 9



C_5

Figure 10

To obtain upper and lower bounds for $q^*(G)$, consider a hypergraph $H = (\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m)$ whose vertices are the maximal matchings (with respect to inclusion) $M_1, M_2, \ldots$ of G, and where $\overline{E}_i$ is the set of matchings M containing the edge E_i of G.

Thus $E_i \cap E_j = \emptyset$ if and only if $\overline{E}_i \cap \overline{E}_j \neq \emptyset$. A minimum transversal T of H defines an optimal colouring of the edges of G, each point of T defining a matching of G in which we colour the edges with the same colour. A minimum k-transversal $t(M)$ of H defines an optimal colouring of kG with $\Sigma t(M_i)$ colours, each matching M_i corresponding to a set of $t(M_i)$ distinct colours. Thus

$$m(H) = m(G)$$

$$\tau(H) = q(G)$$

$$\tau_k(H) = q(kG)$$

$$\tau^*(H) = q^*(G)$$

$$\Delta(H) = \nu(G)$$

If we further denote by $\Delta_0(G)$ the maximum number of pairwise intersecting edges of G (constituting either a "star" or a "multiple triangle") we also have

$$\nu(H) = \Delta_0(G)$$

Theorems 1 and 12 yield:

$$\Delta_0(G) \leq \max_{G' \subseteq G} \frac{m(G')}{\nu(G')} \leq q^*(G) \leq q(G) \leq (1 + \log \nu(G)) q^*(G).$$

These inequalities may be made more precise by studying the family $\mathcal{A}$ of subsets A of X with $|A| \geq 3$ and $|A|$ odd. Indeed, for every $A \in \mathcal{A}$ we have

$$q^*(G) \geq \max_{G' \subseteq G} \frac{m(G')}{\nu(G')} \geq \frac{m(G_A)}{\nu(G_A)} \geq \frac{m(G_A)}{\frac{1}{2}(|A|-1)}$$

Moreover,

$$q^*(G) \geq \Delta_0(G) \geq \Delta(G)$$

We thus have

$$(1) \quad q^*(G) \geq \max \left\{ \Delta(G);\ \max_{A \in \mathcal{A}} \frac{2m(G_A)}{|A|-1} \right\}$$

It can be shown (Seymour [1978]) that we have equality in (1) for every multigraph G.

5. Ryser's Conjecture

We now complete our study of the relationship between the coefficients $\tau^*(H)$, $\nu(H)$ and $\tau(H)$. In the case $r = 2$, Corollary 3 of Theorem 5 can be reformulated as follows:

Theorem 13: *Let G be an r-uniform hypergraph with $r = 2$. Then*

$$(0) \quad \tau^*(G) \leq \frac{3}{2}\,\nu(G) = \frac{r^2-r+1}{r}\,\nu(G)$$

Further we have equality in (0) if and only if G is the union of pairwise disjoint triangles.

In the case $r > 2$ we have an analogous result:

Theorem 14 (Furedi [1981]). *Let H be an r-uniform hypergraph, $r \geq 3$. Then*

$$(1) \quad \tau^*(H) \leq \frac{r^2-r+1}{r}\,\nu(H).$$

Equality in (1) is attained if and only if H is the union of pairwise disjoint projective planes of rank r. Further, if H does not contain $p+1$ pairwise disjoint projective planes of rank r then

$$(2) \quad \tau^*(H) \leq (r-1)\,\nu(H) + \frac{p}{r}$$

Observe first that if H is the union of k projective planes P_r of rank r, we have $\nu(H) = k$; from Theorem 7, $\tau^*(P_r) = \dfrac{n(P_r)}{r} = \dfrac{r^2-r+1}{r}$, so

$$\tau^*(H) = \frac{r^2-r+1}{r}\,k = \frac{r^2-r+1}{r}\,\nu(H)$$

Observe also that for $r = 2$, the statement equivalent to (2) is not valid, since $\tau^*(C_5) = 2.5 \neq (r-1)\,\nu(C_5)$.

Corollary 1. *Let H be an intersecting r-uniform hypergraph. Then*

$$(4) \quad \Delta(H) \geq \frac{r}{r^2-r+1}\, m(H).$$

Equality holds in (4) if and only if H is the graph K_3 or a projective plane of rank $r \geq 3$.

Proof: Since $\nu(H) = 1$ we have, from theorems 13 and 14,

$$\frac{m(H)}{\Delta(H)} \leq \tau^*(H) \leq \frac{r^2-r+1}{r}.$$

If H is a K_3 or a projective plane of rank $r \geq 3$ we have, from Theorem 7,

$$\tau^*(H) = \frac{n}{r} = \frac{m(H)}{\Delta(H)},$$

whence

$$\frac{m(H)}{\Delta(H)} = \frac{r^2-r+1}{r}.$$

In every other case, the inequality in (4) is strict (from Theorems 13 and 14).

Corollary 2. *If H is a regular r-uniform hypergraph of order n then*

$$(5) \quad \nu(H) \geq \frac{n}{r^2-r+1}.$$

Equality holds in (5) if and only if H is the union of $\nu(H)$ disjoint projective planes of rank r (if $r \geq 3$) or $\nu(H)$ disjoint triangles (if $r = 2$).

Proof. From Theorem 7, we have

$$\tau^*(H) = \frac{n}{r}$$

and the result follows directly from theorems 13 and 14. Corollary 2 was conjectured by Bollobás-Erdős, proved in the case $r = 2$ by Bollobás-Eldridge [1976].

Corollary 3. *Let H be an r-uniform hypergraph, $r \geq 3$, which contains no projective plane of rank r as a partial subhypergraph. Then*

$$\tau^*(H) \leq (r-1)\nu(H)$$

This inequality is satisfied in particular for those values of r such that no projective plane of rank r exists (e.g. $r = 7$).

Corollary 4. *Let H be an r-uniform hypergraph whose vertex set is the disjoint union of sets $X^1, X^2, \ldots, X^r$, and whose edges E satisfy $|E \cap X^i| = 1$ for all i ("r-partite hypergraph"). Then*

$$\tau^*(H) \leq (r-1)\nu(H)$$

Proof. For $r = 2$, H is a bipartite graph, which implies

$$\tau^*(H) = \tau(H) = \nu(H) = (r-1)\nu(H)$$

For $r \geq 3$ we have $H \subset K^r_{n_1, n_2, \ldots, n_r}$; it is easy to check that the complete r-partite hypergraph contains no projective plane of rank r, thus the same is true of H, and corollary 3 gives

$$\tau^*(H) \leq (r-1)\nu(H)$$

Observe that for a bipartite graph G, König's theorem implies the stronger inequality

$$\tau(G) \leq (r-1)\nu(G).$$

This observation prompted Ryser [1970] to conjecture the following:

Ryser's Conjecture. *Every r-partite hypergraph H satisfies*

$$\tau(H) \leq (r-1)\nu(H).$$

Remark: Theorems 13 and 14 were used by Frankl and Füredi [1983] to give an upper bound for $\dfrac{m(H)}{\Delta(H)}$ as a function of r, and hence to generalise a theorem of Chvátal and Hansen [1976] (case $r = 2$) and a theorem of Bollobás [1977] (case $r = 3$).

6. Transversal Number of Product Hypergraphs

Given a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on a set X and a hypergraph $H' = (F_1, F_2, \ldots, F_{m'})$ on a set Y, define their product to be the hypergraph $H \times H'$ whose vertices are the elements of the cartesian product $X \times Y$, and whose edges are the sets $E_i \times F_j$ with $1 \leq i \leq m$, $1 \leq j \leq m'$. The order of $H \times H'$ is $n(H \times H') = n(H)n(H')$, the rank is $r(H \times H') = r(H)r(H')$.

Numerous combinatorial problems arise concerning the coefficients ν, τ or χ of product hypergraphs.

Example 1: Polarised partitions (Erdős, Rado [1956]). Consider the set of points (x,y) in the plane with integer coordinates $1 \leq x \leq p$, $1 \leq y \leq q$. What is the largest integer $P(p,q,r,s)$ such that in every colouring of these points with $P(p,q,r,s)$ colours there exist rs points lying in r columns and s rows, each having the same colour? If K_p^r denotes the complete r-uniform hypergraph on p points, $P(p,q,r,s)$ is just $\chi(K_p^r \times K_q^s)-1$ where $\chi(H)$ is the chromatic number of H (cf. chapter 4). For example, $\chi(K_4^2 \times K_6^2) = 2$, and a 2-colouring of the hypergraph with colours 0 and 1 is given in the following figure:

$$4\left\{\underbrace{\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}}_{6}\right.$$

There is no 2×2 submatrix whose entries are all equal.

Thus $P(6,4,2,2) = \chi(K_4^2 \times K_6^2)-1 = 2-1 = 1$.

The numbers $\chi(K_p^r \times K_q^s)$ have been studied notably by Erdös and Rado [1856], Chvátal [1969], Reiman [1958] and Sterboul [1972] [1983].

Example 2: Zarankiewicz numbers [1951]. In 1951, Zarankiewicz posed the following problem: what is the smallest integer z such that every 0,1 matrix with q rows and p columns, with z entries equal to 1, necessarily contains a submatrix with s rows and r columns each of whose entries is 1? This number $Z(p,q,r,s)$, called the *Zarankiewicz number* is the subject of an abundant literature (cf. Guy [1969], Sterboul [1983]). If $\alpha(H)$ is the *stability number* of a hypergraph H, i.e. the largest number of vertices which contain no edge of H (cf. chapter 4), we have

$$Z(p,q,r,s) = \alpha(K_p^r \times K_q^s)+1 = pq+1-\tau(K_p^r \times K_q^s)$$

Example 3. (Hales [1973]): What is the least number of points in the rectangle of points (x,y) having integer coordinates $1 \leq x \leq p$, $1 \leq y \leq q$, such that each unit square contains at least one of these points? If P_n denotes the graph whose vertices are the integers $1,2,\ldots,n$, with x,y adjacent if and only if $|x-y| = 1$, then the answer is

$$\tau(P_p \times P_q) = [\frac{p}{2}][\frac{q}{2}].$$

Theorem 15. *For two hypergraphs $H = (E_1,E_2,\ldots,E_m)$ and $H' = (F_1,F_2,\ldots,F_{m'})$ on X and Y respectively we have*

$$\nu(H)\nu(H') \le \nu(H\times H') \le \tau^*(H)\nu(H') \le \tau^*(H)\tau^*(H')$$
$$= \tau^*(H\times H') \le \tau^*(H)\tau(H') \le \tau(H\times H') \le \tau(H)\tau(H').$$

Proof:

1. If $\{E_i / i \in I\}$ and $\{F_j / j \in J\}$ are two maximum matchings of H and H' respectively, then, for $(i,j),(i',j') \in I \times J$, $(i,j) \ne (i',j')$, we have

$$(E_i \times F_j) \cap (E_{i'} \times F_{j'}) = \varnothing.$$

Thus $\{E_i \times F_j,\ i \in I,\ j \in J\}$ is a matching of $H \times H'$, whence

$$\nu(H)\nu(H') = |I||J| \le \nu(H\times H')$$

2. If $\{E_i \times F_j / (i,j) \in K\}$ is a maximum matching of $H \times H'$, the function

$$z(E_i) = \frac{1}{\nu(H')} \left|\{j / (i,j) \in K\}\right|$$

constitutes a fractional matching of H, since

$$\sum_{E\in H(x)} z(E) = \frac{1}{\nu(H')} \left|\{E_i\times F_j / E_i \in H(x); (i,j)\in K\}\right| \le \frac{\nu(H')}{\nu(H')} = 1$$

Hence

$$\nu(H\times H') = |K| = \sum_i z(E_i)\nu(H') \le \tau^*(H)\nu(H')$$

3. We have $\tau^*(H)\nu(H') \le \tau^*(H)\tau^*(H')$ from Theorem 1.

4. Let $q(E)$ and $q'(F)$ be fractional matchings for H and H' respectively. The function $z(E\times F) = q(E)q'(F)$ is a fractional matching of $H \times H'$, since

$$\sum_{\substack{E\in H(x)\\ F\in H'(y)}} z(E\times F) = \sum_{E\in H(x)} q(E) \sum_{F\in H'(y)} q'(F) \le 1.$$

Thus $\tau^*(H\times H') \ge \sum_{i,j} z(E_i\times F_j) = \sum_i q(E_i)\sum_j q'(F_j) = \tau^*(H)\tau^*(H')$.

We now show the reverse inequality. Let $t(x)$ and $t'(y)$ be optimal fractional transversals for H and H' respectively. The function $p(x,y) = t(x)t'(y)$ is a fractional transversal of $H \times H'$, since

$$\sum_{(x,y)\in E_i\times F_j} p(x,y) = \sum_{x\in E_i} t(x) \sum_{y\in F_j} t'(y) \ge 1$$

Thus

$$\tau^*(H \times H') \leq \sum_{x,y} p(x,y) = \sum_x t(x) \sum_y t'(y)$$
$$= \tau^*(H)\tau^*(H')$$

Thus $\tau^*(H \times H') = \tau^*(H)\tau^*(H')$.

5. We have $\tau^*(H \times H') = \tau^*(H)\tau^*(H') \leq \tau^*(H)\tau(H')$ from Theorem 1.

6. As for 2, we may show that

$$\tau^*(H)\tau(H') \leq \tau(H \times H').$$

7. As for 1, we may show that

$$\tau(H \times H') \leq \tau(H)\tau(H').$$

Corollary (McEliece, Posner [1971]). *Every hypergraph H satisfies*

$$\tau^*(H) = \lim_{k \to \infty} \sqrt[k]{\tau(H^k)},$$

where $H^k = H \times H \times \cdots \times H$ is the product of k terms equal to H.

Proof. From Theorem 12 we may write

$$\tau^*(H)^k = \tau^*(H^k) \leq \tau(H^k) \leq [1 + \log \Delta(H^k)]\tau^*(H^k)$$
$$\leq [1 + k \log \Delta(H)]\tau^*(H)^k.$$

It is easy to see that $(1 + k \log \Delta(H))^{1/k} \to 1$ as $k \to \infty$, giving the desired result.

The following results, sharpening the statement of Theorem 15 are due to Berge and Simonovits [1972].

Theorem 16. *Every hypergraph H satisfies*

$$\tau^*(H) = \min_{H'} \frac{\tau(H \times H')}{\tau(H')}$$

Proof. From Theorem 15, we have

$$\tau^*(H) \leq \min_{H'} \frac{\tau(H \times H')}{\tau(H')}$$

We shall show the reverse inequality. There exists an integer k such that $\tau^*(H) = \dfrac{\tau_k(H)}{k}$. Let $t(x)$ be an optimal k-transversal for H; consider a set Y of cardinality $p = \tau_k(H)$, and a partition $(Y_1, Y_2, \ldots, Y_n)$ of Y with $|Y_i| = t(x_i)$ for each i.

The complete $(p-k+1)$-uniform hypergraph $H' = K_p^{p-k+1}$ on Y satisfies $\tau(H') = k$. Consider the set

$$\bar{T} = \bigcup_{i=1}^{n} (\{x_i\} \times Y_i)$$

For each edge E of H,

$$|(E \times Y) \cap \bar{T}| = \sum_{x \in E} t(x) \geq k$$

Further, since each edge F of H' is of cardinality $p-k+1$, the set $E \times F$ meets $\bar{T}$, by the pigeonhole principle; thus $\tau(H \times H') \leq |\bar{T}| = |Y| = \tau_k(H)$. Hence

$$\frac{\tau(H \times H')}{\tau(H')} \leq \frac{\tau_k(H)}{k} = \tau^*(H).$$

Theorem 17. *Every hypergraph H with the Helly property satisfies*

$$\tau^*(H) = \max_{H'} \frac{\nu(H \times H')}{\nu(H')}$$

Proof. From Theorem 15 we have

$$\tau^*(H) \geq \max \frac{\nu(H \times H')}{\nu(H')}$$

We shall show the reverse inequality. There exists an integer s such that

$$\tau^*(H) = \frac{\nu_s(H)}{s}$$

Let $H_0 = (E_k / k \in K)$ be a maximum s-matching of H; thus $|K| = \nu_s(H)$ and $\Delta(H_0) = s$.

Let Y be the set of maximum matchings of the hypergraph H_0; for $k \in K$ let F_k be the set of maximal matchings of H_0 which contain E_k. The hypergraph $H' = \{F_k / k \in K\}$ on Y satisfies $\nu(H') \leq \Delta(H_0) = s$, since H has the Helly property. The hypergraph $H \times H'$ admits $\{E_k \times F_k / k \in K\}$ as a matching, since $(E_k \times F_k) \cap (E_{k'} \times F_{k'}) \neq \emptyset$ implies both $E_k \cap E_{k'} \neq \emptyset$ and $F_k \cap F_{k'} \neq \emptyset$, which is a contradiction. Thus

$$\nu(H \times H') \geq |K| = \nu_s(H) = s\tau^*(H).$$

Hence

$$\frac{\nu(H \times H')}{\nu(H')} \geq \frac{s\tau^*(H)}{s} = \tau^*(H).$$

Q.E.D.

We may relate the product of hypergraphs to the numbers $P(p,q,2,2) = \chi(K_p \times K_q)-1$ defined in example 1, and to the *Ramsey number* $R(p,q)$ (the least integer m such that every 2-colouring of the edges of K_m contains either a p-clique of the first colour or a q-clique of the second). It is known that $R(p,q) \leq \binom{p+q-2}{p-1}$, but, with the exception of a few particular cases, the exact value of $R(p,q)$ is not known. Recall that the chromatic number $\chi(H)$ is the least number of colours necessary to colour the vertices of H such that no edge is monochromatic (except for loops).

Theorem 18. *We have*

$$\max_{\substack{\chi(H) \leq p \\ \chi(H') \leq q}} \chi(H \times H') = \chi(K_p \times K_q)$$

the maximum being taken over hypergraphs H, H' without loops.

Proof. Let $H = (E_i)$ be a hypergraph without loops on X with $\chi(H) \leq p$, and let $H' = (F_j)$ be a hypergraph without loops on Y with $\chi(H') \leq q$. On H we have a p-colouring $g(x) \in \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and on H' we have a q-colouring $g'(y) \in \{\beta_1, \beta_2, \ldots, \beta_q\}$. We shall show that we may obtain from these a colouring of $H \times H'$ with $\chi(K_p \times K_q)$ colours. Let K_p be a complete graph on $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and K_q a complete graph on $\{\beta_1, \beta_2, \ldots, \beta_q\}$; colour the vertices of $K_p \times K_q$ with an optimal colouring $F(\alpha, \beta) \in \{1, 2, \ldots, \chi(K_p \times K_q)\}$. Thus four vertices $\alpha_j\beta_j, \alpha_j\beta_k, \alpha_k\beta_j, \alpha_k\beta_k$ are never all with the same colour. Let $\Phi(x,y) = F(g(x), g'(y))$. Since there exist $x_1, x_2 \in E_i$ and $y_1, y_2 \in F_j$ such that $g(x_1) \neq g(x_2)$, $g'(y_1) \neq g'(y_2)$ (since $|E_i| > 1$, $|F_j| > 1$), the set $E_i \times F_j$ is not monochromatic in Φ. Hence Φ is a colouring of $H \times H'$ in $\chi(K_p \times K_q)$ colours, whence $\chi(H \times H') \leq \chi(K_p \times K_q)$. Since this inequality is an equality when $H = K_p$, $H' = K_q$ we have the theorem.

Theorem 19 (Erdös, McEliece, Taylor [1971], anticipated by Hedrlin [1966]). *We have*

$$\max_{\substack{\nu(H) \leq p \\ \nu(H') \leq q}} \nu(H \times H') = R(p+1, q+1)-1$$

where the $R(p,q)$ are the Ramsey numbers.

Proof. 1. We shall show first that $\nu(H) \leq p$, $\nu(H') \leq q$ implies $\nu(H \times H') \leq R(p+1,q+1)-1$. Suppose that this inequality fails for $H = (E_i)$ and $H' = (F_j)$. Put:

$$m = \nu(H \times H') \geq R(p+1,q+1).$$

Let $\{E_i \times F_j / (i,j) \in M\}$ be a maximum matching of $H \times H'$, with $|M| = m$. Consider the complete graph K_m on M: colour the edge $[(i,j),(i',j')]$ red if $E_i \cap E_{i'} = \emptyset$ and blue if $E_i \cap E_{i'} \neq \emptyset$ (and then $F_j \cap F_{j'} = \emptyset$). Since $|M| = m \geq R(p+1,q+1)$ this colouring of the edges of K_m contains either a red $(p+1)$-clique (and then $\nu(H) > p$) or a blue $(q+1)$-clique (and then $\nu(H') > q$); in each case we have a contradiction.

2. Consider the complete graph K_m on $M = \{1,2,...,m\}$ where $m = R(p+1,q+1)-1$. From the definition of Ramsey numbers, there exists a 2-colouring of the edges of K_m forming two partial graphs G, G' with $\omega(G) \leq q$ and $\omega(G') \leq p$.

The dual hypergraph $H = G^*$ of the graph G has edges of the form: E_i ={edges of G incident to vertex i of G}; thus

$$\nu(H) = \omega(\overline{G}) = \omega(G') \leq p.$$

Similarly $H' = (G')^*$ has edges of the form: F_j = {edges of G' incident to vertex j of G'}; thus $\nu(H') \leq q$. From part 1 of the proof, this implies

$$\nu(H \times H') \leq R(p+1,q+1) - 1.$$

For two distinct indices $i,j \in M$ the sets $E_i \times F_i$ and $E_j \times F_j$ are disjoint, so the product hypergraph $H \times H'$ admits $\{E_i \times F_i / i \in M\}$ as a matching, whence

$$\nu(H \times H') \geq |M| = R(p+1,q+1)-1.$$

Thus $\nu(H \times H') = R(p+1,q+1)-1$, and the statement of the theorem follows.

Application: Shannon capacity of a graph.

Define the *normal product* of two simple graphs $G = (X,E)$, $G' = (Y,F)$ to be the graph $G \times G'$ on $X \times Y$ where two vertices (x,y) and (x',y') are adjacent if and only if

$x = x'$ and $[x,y'] \in F$,

or $[x,x'] \in E$ and $y = y'$,

or $[x,x'] \in E$ and $[y,y'] \in F$.

Shannon was interested in the study of the stability number of the normal product of graphs. Indeed, if G is the graph of confusion at reception for a set of signals X and G' the graph of confusion for a set of signals Y, then $\overline{\alpha}(G \times G')$ represents the greatest number of words xy with $x \in X$, $y \in Y$ which cannot be confused at reception. We may also consider words of k signals (taken from X) which form a code; the largest possible number of distinguishable words is then $\overline{\alpha}(G^k)$, where $G^k = G \times G \times \cdots \times G$ is the normal product of k terms equal to G.

Shannon proposed the term *capacity* for the number

$$\max_k \sqrt[k]{\overline{\alpha}(G^k)} = c(G).$$

It is immediate that for all k

$$\overline{\alpha}(G) \leq \sqrt[k]{\overline{\alpha}(G^k)} \leq c(G) \leq \sqrt[k]{\theta(G^k)} \leq \theta(G).$$

The number $c(G)$ is difficult to calculate (Lovasz proved in 1979 that $c(C_5) = \sqrt{5}$).

Let $H(G)$ be the hypergraph formed by the maximal cliques of G, and let $\overline{H}(G)$, or more simply $\overline{H}$, be the dual of $H(G)$. Then

$$n(G) = m(\overline{H})$$

$$\omega(G) = \Delta(\overline{H})$$

$$\overline{\alpha}(G) = \nu(\overline{H})$$

$$\theta(G) = \tau(\overline{H})$$

The minimum value of a q-covering of G by cliques is

$$\theta_q(G) = \tau_q(\overline{H})$$

Also

$$\overline{\alpha}(G^k) = \nu[\overline{H}(G^k)] = \nu(\overline{H}^k)$$

$$\theta(G^k) = \tau(\overline{H}(G^k)] = \tau(\overline{H}^k)$$

Clearly, $\sqrt[k]{\overline{\alpha}(G^k)} \rightarrow c(G)$. The corollary to Theorem 15 shows that we also have

$$\sqrt[k]{\theta(G^k)} \rightarrow \tau^*(\overline{H}).$$

Exercises on Chapter 3

Exercise 1 (§1)

Show that $\frac{\tau_k(H)}{k} \rightarrow \tau^*(H)$.

Hint: use the theorem of Fekete that states that if a series (u_k) is subadditive, i.e. $u_{k+h} \leq u_k + u_h$, then $\frac{u_k}{k} \rightarrow inf \frac{u_k}{k}$.

Exercise 2 (§1)

Show similarly that $\frac{\nu_k(H)}{k} \rightarrow \tau^*(H)$.

Exercise 3 (§1)

Show that if $\frac{\tau_k(H)}{k} = \tau(H)$ for some integer k, then every integer $p \leq k$ satisfies $\frac{\tau_p(H)}{p} = \tau(H)$.

Exercise 4 (§1)

Show that if $\frac{\tau_k(H)}{k} = \tau^*(H)$ for an integer k, then $\frac{\tau_{ks}(H)}{ks} = \tau^*(H)$ for every integer s.

Exercise 5 (§3)

Let X be a finite set of points on a line, and let H be an interval hypergraph on X. Show that H is regularisable if and only if there do not exist two distinct points $x,y \in X$ such that $H(x) \subset H(y)$ and $H(x) \neq H(y)$.

Exercise 6 (§3)

Let H be an r-uniform hypergraph such that the distinct $I_x = \bigcap_{E \in H(x)} E$ form a partition of X, and every edge meeting I_x contains I_x. Show that if $H - H(x)$ is quasi-regularisable for each x, then H is regularisable.

(Berge [1978]; Pulleyblank [1977] in the case of a graph).

Exercise 7 (§3)

Let H be an r-uniform hypergraph without vertices of degree 1, and such that each edge meets at least r other edges of H. Show that the graph $L(H)$ is regularisable.

(Berge [1978]).

Exercise 8 (§3)

Let G be a connected nonbipartite regularisable graph. Show that every graph which admits G as a partial graph is also regularisable.

Hint: use condition (3) of Theorem 10.

Exercise 9 (§6)

Let H be an r-uniform hypergraph of order n, with m edges, regularisable, linear, and containing no projective plane of order r as a partial subhypergraph. Show that

$$\nu(H) \geq \frac{m}{n-1}.$$

In this case we have a better bound than that of Seymour (Theorem 8, Chapter 2).

Exercise 10

Aharoni, Erdős and Linial [1987] have proved that every hypergraph H satisfies

$$\nu(H) \geq \frac{[\tau^*(H)]^2}{m(H)}$$

Check that this interesting inequality holds for some of the hypergraphs described in the examples of Chapter 2, §4 which do not satisfy the König property.

Chapter 4

Colourings

1. Chromatic Number

Let $H = (E_1, E_2, \ldots, E_n)$ be a hypergraph and let k be an integer ≥ 2. A *k-colouring* (of the vertices) is a partition $(S_1, S_2, \ldots, S_k)$ of the set of vertices into k classes such that every edge which is not a loop meets at least two classes of the partition; that is to say

$$E \in H,\ |E| > 1 \Rightarrow E \not\subset S_i \qquad (i = 1,2,\ldots,k).$$

A vertex in S_i will be said to be a "vertex of colour i", and S_i ("the colour set i") may possibly be empty; the only "monochromatic" edges are therefore the loops. For a hypergraph H its *chromatic number* $\chi(H)$ is the smallest integer k for which H admits a k-colouring.

Example: If H is the hypergraph whose vertices are the different waste products in a chemical production factory, and in which the edges are the dangerous combinations of these waste products, the chromatic number of H is the smallest number of waste disposal sites that the factory needs in order to avoid any hazardous situation.

We note that if the hypergraph H is a graph, the chromatic number of H coincides exactly with the usual chromatic number.

For a hypergraph H on X, a set $S \subset X$ is said to be *stable* if it does not contain any edge E with $|E| > 1$. The *stability number* $\alpha(H)$ of H is the maximum cardinality of a stable set of H.

Example: The projective plane on seven points is a hypergraph P_7 with $\alpha(P_7) = 4$, as can be verified immediately from Figure 2 of Chapter 2. We see also that $\chi(P_7) = 3$.

Proposition 1. *Every hypergraph H of order n satisfies $\chi(H)\alpha(H) \geq n$.*

Proof. Let us consider a k-colouring $(S_1, S_2, \ldots, S_k)$ of H in $k = \chi(H)$ colours; we have

$$n = \sum_{i=1}^{k} |S_i| \leq k\alpha(H) = \chi(H)\alpha(H).$$

This gives the stated inequality.

Proposition 2. *Every hypergraph H of order n satisfies* $\chi(H) + \alpha(H) \leq n+1$.

Proof. Let S be a maximum stable set of H. We can colour all the vertices of S with a first colour, and use $n-\alpha(H)$ other colours to colour, each with a different colour, the vertices of $X-S$. From this

$$\chi(H) \leq (n-\alpha(H)) + 1.$$

This gives the stated inequality.

We call a *β-star of a vertex x* a family $H^\beta(x) \subset H(x)$ such that

(i) $E \in H^\beta(x) \Rightarrow |E| \geq 2.$

(ii) $E, E' \in H^\beta(x) \Rightarrow E \cap E' = \{x\}.$

We call the *β-degree* of a vertex x the largest number of edges of a β-star of x. We denote by $d_H^\beta(x)$ the β-degree of x, by $\Delta^\beta(H) = \max_{x \in X} d_H^\beta(x)$ the maximum β-degree, and by $\delta^\beta(H)$ the minimum β-degree. H/A denotes as usual the family of edges of H contained in A; then we can obtain upper bounds for the chromatic number with the following assertion:

Theorem 1. *Every hypergraph H on X satisfies*

$$\chi(H) \leq \max_{A \subset X} \delta^\beta(H/A) + 1.$$

Proof. Let $p = \max \delta^\beta(H/A)$. We shall seek to colour the vertices of H successively in increasing order of their indices using only $p+1$ colours. Let us index the vertices in the order $x_n, x_{n-1}, \ldots, x_1$ by the following rule:

(i) x_n is a vertex of minimum β-degree in H;

(ii) for $i < n$, x_i is a vertex whose β-degree in $H/X - \{x_{i+1}, x_{i+2}, \ldots, x_n\}$ is $\leq p$.

Suppose that we have coloured $x_1, x_2, \ldots, x_{i-1}$ with the colours $1, 2, \ldots, p+1$ without any edge of H being completely coloured and monochromatic. The star $H(x_i)$ does not contain $p+1$ edges containing only coloured vertices (except for x_i), monochromatic, and bearing respectively the colours 1,2,... and $p+1$; for such a set of edges would constitute a β-star with $p+1$ edges, which contradicts the rule for choosing x_i. Thus there exists a colour $j \leq p+1$ which we can attach to x_i without any edge becoming completely coloured and monochromatic. Thus, step by step, we colour all the vertices with $p+1$ colours.

Corollary 1 (Lovász [1968]). *For every hypergraph H of maximum β-degree Δ^β, we have $\chi(H) \leq \Delta^\beta(H) + 1$. Moreover, for every rank r, this bound is the best possible, since $\chi(K_n^r) = \Delta^\beta(K_n^r) + 1$.*

Indeed, let $q = \Delta^\beta(H) + 1$. The set of vertices x with $d_H^\beta(x) \geq q$ being empty, Theorem 1 gives: $\chi(H) \leq q$. Moreover, we have

$$\chi(K_n^r) = [\frac{n}{r-1}]^* \geq \frac{n}{r-1} \geq \frac{(r-1)\Delta^\beta(K_n^r)+1}{r-1} = \Delta^\beta(K_n^r) + \frac{1}{r-1}.$$

Thus we have $\chi(K_n^r) = \Delta^\beta(K_n^r) + 1$.

Corollary 2. *For every hypergraph H of order n*

$$\alpha(H) \geq \frac{n}{\Delta^\beta(H)+1}.$$

For Proposition 1 shows that

$$\alpha(H) \geq \frac{n}{\chi(H)} \geq \frac{n}{\Delta^\beta(H)+1}.$$

Corollary 3. *For every hypergraph H of order n without loops*

$$\tau(H) \leq \frac{n\Delta^\beta(H)}{\Delta^\beta(H)+1}.$$

For the complement of a stable set being a transversal, we have

$$\tau(H) = n - \alpha(H) \leq \frac{n\Delta^\beta(H)}{\Delta^\beta(H)+1}.$$

From this the stated inequality follows.

These corollaries enable us to solve easily a large number of combinatorial problems.

Application 1. Given a simple graph G on X of maximum degree h, what is the smallest number of colours necessary to colour the vertices such that no cycle is monochromatic? (Motzkin [1968]).

Let us consider a hypergraph H on X whose edges are the elementary cycles of G. The answer is then, from Corollary 1,

$$\chi(H) \leq \Delta^{\beta}(H)+1 \leq [\frac{h}{2}] + 1.$$

Application 2. Given a simple graph G on X of maximum degree h, what is the smallest number of colours necessary to colour the vertices such that every subgraph G_i induced by a colour i has maximum degree $< t$? (Gerencser [1965]).

This number $\gamma_t(G)$ generalizes the usual chromatic number (the case $t = 1$); if H is the hypergraph on X whose edges are the subgraphs of maximum degree t, then Corollary 1 gives:

$$\gamma_t(G) = \chi(H) \leq \Delta^{\beta}(H) + 1 \leq [\frac{h}{t}] + 1.$$

Application 3. Given a simple graph G on X, what is the smallest number of colours necessary to colour the vertices such that no elementary path of length k is monochromatic? (Chartrand, Geller, Hedetniemi [1968]). This number $\bar{\gamma}_k(G)$ generalizes the usual chromatic number (the case $k = 1$); it is also the chromatic number of a hypergraph H defined in an obvious manner, giving immediately an upper bound.

Application 4. Given a simple graph G on X what is the smallest number of colours necessary to colour the vertices of G such that no clique of size k is monochromatic? (Sachs, Schaüble [1967]).

This number $\bar{\bar{\gamma}}_k(G)$ generalizes the usual chromatic number (the case $k = 2$); it is also the chromatic number of a hypergraph H defined in an obvious manner, which leads immediately to an upper bound.

Application 5. Symmetric Ramsey Numbers.

We consider the complete graph K_n, and propose to associate with each of its edges one of the colours $1,2,\ldots,q$ in such a way that no clique of p elements of K_n has all its edges of the same colour. The smallest integer n for which this association is impossible is called the *(symmetric) Ramsey Number* and is denoted by $R(p,p,...,p)$, or R_p^q. In other words, if $n < R_p^q$, there exists an association of colours $1,2,...,q$ with the edges of K_n such that no K_p has all its edges of the same colour.

We can apply Theorem 1 to this problem if we define a hypergraph on the set of edges of K_n, denoted K_n/K_p, whose edges are all the sets of edges of K_n which induce a K_p. Indeed $n \leq R_p^q - 1$ is equivalent to saying that the hypergraph K_n/K_p is q-colourable.

Let us consider for example the case $p = 3$. Then $n < R_3^q$ is equivalent to saying that K_n can be decomposed into q graphs without triangles. We know that K_5 can be decomposed into two graphs without triangles, in fact two pentagons. We know also that K_{16} can be decomposed into three graphs without triangles; one manner of doing this is due to Greenwood and Gleason [1955], the other to Kalbfleisch and Stanton [1968]. Finally, we know also that K_{64} can be decomposed into four graphs without triangles (cf. Graham [1965], Chung [1973]). Thus

(1) $$R_3^2 \geq 6; \quad R_3^3 \geq 17; \quad R_3^4 \geq 65.$$

On the other hand,

(2) $$R_3^q \leq 1 + q! \sum_{k=0}^{q} \frac{1}{k!}.$$

Indeed, let K be a complete graph of order $R_3^q - 1$ which is decomposed into q graphs without triangles $G_1,G_2,...,G_q$, let a be a vertex of K, and let A_i be the set of vertices of K adjacent to a in G_i. As the subgraph K_{A_i} does not contain any edge of G_i (since G_i is without triangles), it is decomposible into $q-1$ graphs without triangles, whence thus

$$|A_i| \leq R_3^{q-1} - 1.$$

We deduce from this that

$$R_3^q - 1 = 1 + d_k(a) = 1 + \sum_{i=1}^{q} |A_i| \leq 1 + (R_3^{q-1} - 1)q.$$

This recurrence formula gives immediately (2).

Together (1) and (2) give:

$$R_3^2 = 6;\; R_3^3 = 17;\; 65 \leq R_3^4 \leq 66.$$

Theorem 2 (Lepp Gardner [1973]). *Let H be a linear hypergraph without loops. Then $\chi(H) \leq \Delta(H)$, except for the two following cases:*

(i) $\Delta(H) = 2$, *and a connected component of H is a graph consisting of an odd cycle;*

(ii) $\Delta(H) > 2$ *and a connected component of H is the complete graph of order* $\Delta(H) + 1$.

In these two cases we have $\chi(H) = \Delta(H) + 1$.

If H is linear, we have $\Delta^\beta(H) = \Delta(H)$, and Theorem 1 gives: $\chi(H) \leq \Delta(H) + 1$.

It follows from a theorem of Lepp Gardner [1977] that this inequality is strict when H is linear and does not satisfy (i) or (ii).

This result is an extension of Brooks's Theorem (see *Graphs*, Theorem 6, Chapter 15).

2. Particular Kinds of Colourings

Besides the concept of colouring defined in the preceding paragraph - often called "weak" colouring - there exist other concepts which generalize to hypergraphs that of the colouring of a graph.

Strong colourings. For a hypergraph H on X a *strong k-colouring* (of the vertices) is a k-partition $(S_1,S_2,...,S_k)$ of X such that no colour appears twice in the same edge; that is to say such that for every edge E

$$|E \cap S_i| \leq 1 \qquad (i = 1,2,...,k).$$

The *strong chromatic number* of a hypergraph H, denoted by $\gamma(H)$, is the smallest integer k for which H admits a strong k-colouring. We note that every strong colouring is certainly a colouring, and consequently $\gamma(H) \geq \chi(H)$. However, $\gamma(H)$ is nothing more than the chromatic number of the graph $[H]_2$ (2-section of H); for this reason we shall not study the strong chromatic number for its own sake.

Equitable colourings. For a hypergraph H on X, an *equitable k-colouring* (of the vertices) is a k-partition $(S_1,S_2,...,S_k)$ of X such that in every edge E all the colours appear the same number of times (or to within 1, if k does not divide $|E|$); that is to say:

$$\left[\frac{|E|}{k}\right] \leq |E \cap S_i| \leq \left[\frac{|E|}{k}\right]^* \qquad (i = 1,2,...,k)$$

We note that an equitable k-colouring is certainly a k-colouring. Furthermore every strong k-colouring is an equitable k-colouring. The equitable colourings of a hypergraph will be studied more particularly for unimodular hypergraphs (§2, Chapter 5).

Good colourings. For a hypergraph H on X a *good k-colouring* is a k-partition $(S_1,S_2,...,S_k)$ of X such that every edge E contains the largest possible number of different colours (taking account of the value of k), namely

$$\min\{|E|,k\}.$$

We note that a good colouring is certainly a colouring. Moreover, *for* $k = 2$, *a good k-colouring is simply a bicolouring; for* $k \leq \min|E|$, *it is a partition of* X *into* k *transversal sets; for* $k \geq \max|E|$, *it is a strong colouring.* Finally, *for every* k, *an equitable k-colouring is a good colouring.*

Good colourings will be studied particularly for balanced hypergraphs (§ 3, Chapter 5).

I-regular colourings. For a hypergraph H on X, let us associate with every edge E_j two integers a_j and b_j with $0 \leq a_j \leq b_j$, and let $I = \{[a_j,b_j]/j = 1,2,...,m\}$. An *I-regular k-colouring* of H is a k-partition $(S_1,S_2,...,S_k)$ of X such that for every edge E_j

$$a_j \leq |E_j \cap S_i| \leq b_j \qquad (i = 1,2,...,k)$$

We note that an I-regular colouring is also a colouring. Moreover we note:

(1) Every colouring is an I-regular colouring with $a_j = 0$, $b_j = \max\{1, |E_j|-1\}$.

(2) Every strong colouring is an I-regular colouring with $a_j = 0$, $b_j = 1$.

(3) Every equitable colouring is an I-regular colouring with $a_j = \left[\frac{|E_j|}{k}\right]$, $b_j = \left[\frac{|E_j|}{h}\right]^*$.

I-regular colourings were introduced by de Werra [1979] who studied the sequences $s_1 \geq s_2 \geq \cdots \geq s_k$ for which there exists an I-regular k-colouring $(S_1,S_2,...,S_k)$ with $s_1 = |S_1|$, $s_2 = |S_2|$, etc. Some interesting theorems on certain I-regular k-colourings of the edges of a simple graph were obtained by Hilton and Jones [1978].

By way of an exercise one can verify that if $r(H) = 2$, all these definitions give exactly the usual colouring of a graph. We can verify also that if H is an interval hypergraph whose vertices are the points $x_1,x_2,...,x_n$ (in this order) on a line, we obtain an equitable k-colouring of H by using successively the colours $1,2,...,k$, $1,2,...,k$, $1,2,...$ to colour the points from left to right; thus we see that an interval hypergraph has a (weak) chromatic number equal to 2 and a strong chromatic number equal to the rank $r(H)$.

3. Uniform Colourings

For a hypergraph H of order n, a k-colouring $(S_1,S_2,...,S_k)$ is said to be *uniform* if the number of vertices of the same colour is always the same (to within one), that is to say if we have

$$[\frac{n}{k}] \leq |S_i| \leq [\frac{n}{k}]^* \qquad (i = 1,2,...,k).$$

The problem of the existence of a uniform k-colouring arises in numerous scheduling problems.

Example 1. Organizing a colloquium. The organizers of a scientific colloquium have at hand q half-days to organize n sessions $x_1,x_2,...,x_n$, each lasting a half-day. Certain people have to be present at all the sessions of a set $E_1 \subset \{x_1,x_2,...,x_n\}$; others at all those of a set $E_2 \subset \{x_1,x_2,...,x_n\}$, thus defining a hypergraph $H = (E_1,E_2,...,E_m)$ on $\{x_1,x_2,...,x_n\}$. Can one organize the n sessions respecting these constraints with only p conference rooms? It is obviously necessary that $pq \geq n$, that is

$$p \geq [\frac{n}{q}]^*.$$

This condition is necessary and sufficient if the hypergraph H admits a uniform strong q-colouring $(S_1,S_2,...,S_q)$. Indeed in this case the set of sessions taking place during the half-day i may be defined by a set S_i which satisfies:

(1) $$|S_i \cap E_j| \leq 1 \qquad (j = 1,2,...,m),$$

(2) $$|S_i| \leq [\frac{n}{q}]^* \leq p.$$

All the constraints are therefore satisfied.

For the existence of a uniform strong k-colouring we have a well-known theorem of Hajnal and Szemerédi (cf. *Graphs*, Chapter 13, §2), as follows: a graph $[H]_2$ of maximum degree h admits a uniform colouring for every $k \geq h+1$. Therefore H admits a uniform strong k-colouring for every $k \geq h+1$.

(For a simpler proof, see Szemeredi [1975]).

Example 2. Organizing an air show. In the course of an air show an aeroplane takes off every ten minutes and two planes may not be displayed in flight simultaneously. There are m possible buyers who want to be present at these exhibitions at different times, and it is known in advance at what interval of time E_j the buyer j will be present. This defines a hypergraph $H = (E_1,E_2,...,E_m)$ over the set of flight times. Moreover each of the k exhibitors wishes to show his craft in flight to all the buyers and to get the same total exhibition time. It is obviously necessary that $k \leq \min_j |E_j|$. This condition is also sufficient if the hypergraph H admits a uniform good k-colouring $(S_1,S_2,...,S_k)$. Indeed the set of times allocated to the ith exhibitor being defined by the set S_i, all the constraints will be satisfied, for we have

$$S_i \cap E_j \neq \emptyset \qquad (i,j)$$

$$-1 \leq |S_i| - |S_j| \leq 1 \qquad (i,j).$$

We note that the hypergraph H is here an interval hypergraph, and that for every k an interval hypergraph admits a good uniform k-colouring: it is enough to allot successively to the vertices the colours $1,2,...,k$, $1,2,...$ going from left to right along the time axis.

Example 3. Organizing a ping-pong tournament. A set of n players $x_1,x_2,...,x_n$ take part in a tournament where all the matches planned between the players are defined by the m edges of a graph G on $\{x_1,x_2,...,x_n\}$. The duration of a match must not exceed one hour; the tournament has to be finished at the end of p hours, and there are available q ping-pong tables. In order for these constraints to be realized, it is necessary that the maximum degree of G does not exceed p and that $pq \geq m$, that is

(1) $\qquad p \geq \Delta(G)$

(2) $$q \geq [\frac{m}{p}]^*.$$

Conditions (1) and (2) are necessary and sufficient if the edges of G have a uniform p-colouring (a p-colouring of the edges of a graph G being by definition a strong p-colouring of the vertices of the dual hypergraph G^*).

Clearly, if $(E_1,E_2,...,E_p)$ is a uniform p-colouring of the edges of G, the matching E_i defines the matches to be played during the ith hour, since

$$|E_i| \leq [\frac{m}{p}]^* \leq q.$$

We note that McDiarmid [1972] showed that the edges of a graph G admit a uniform k-colouring for every $k \geq \Delta(G)+1$.

Theorem 3. *Let H be a hypergraph which has a good k-colouring. Suppose that for every good k-colouring $(S_i/i \in I)$ and every pair of classes (S_1,S_2) with $|S_2| \geq |S_1| + 2$, the subhypergraph $H_{S_1 \cup S_2}$ admits a bicolouring (S'_1,S'_2) with*

(1) $$|S_1| + 1 \leq |S'_1| \leq |S'_2| \leq |S_2| - 1$$

Then H admits a good k-colouring which is uniform.

Proof. Let $d = \max_{i,j}(|S_i| - |S_j|)$ be the "deficiency" of a good colouring $(S_1,S_2,...,S_k)$ of H. We shall proceed step by step to transform this k-colouring so that it becomes uniform. If $d \leq 1$, the colouring is uniform. If $d \geq 2$, consider two classes, for example S_1 and S_2, with

$$|S_1| = \min |S_i|$$
$$|S_2| = \max |S_i|.$$

As $|S_2| \geq |S_1| + 2$ there exists a bicolouring (S'_1,S'_2) of $H_{S_1 \cup S_2}$ satisfying the inequalities (1). It is easy to verify that $(S'_1,S'_2,S_3,S_4, \ldots ,S_k)$ is again a good k-colouring of H. Moreover, by virtue of (1), we have

$$|S_2'| - |S_1'| \le (|S_2|-1)-(|S_1|+1) = d-2$$

$$|S_1'| - |S_2'| \le 0 \le d-2$$

$$|S_2'| - |S_i| \le |S_2|-|S_i|-1 \le d-1 \qquad (i \ne 1,2)$$

$$|S_1'| - |S_2'| \le |S_1|-|S_i|-1 \le d-1 \qquad (i \ne 1,2)$$

$$|S_i| - |S_2'| \le |S_i|-|S_1|-1 \le d-1 \qquad (i \ne 1,2)$$

$$|S_i| - |S_1'| \le |S_i|-|S_1|-1 \le d-1 \qquad (i \ne 1,2)$$

We have therefore decreased the number of pairs (S_p,S_q) with $|S_p| - |S_q| \ge d$. By repeating this transformation we decrease the deficiency down to $d \le 1$; the good colouring finally obtained is thus uniform. This is what was to be proved.

Corollary 1 (McDiarmid [1972]). *Let G be a multigraph with chromatic index $q(G)$. For $k \ge q(G)$ the edges of G admit a uniform strong k-colouring.*

Indeed every strong k-colouring of the edges with $k \ge q(G) \ge \Delta(G)$ is a good k-colouring. Furthermore the edges having one of the colours i or j make up either even cycles or open paths; if the colour i appears more often than colour j there exists an open path having at each end an edge of colour i; by interchanging the colours on this path we obtain a colouring satisfying (1) which enables us to apply Theorem 3.

Corollary 2 (de Werra [1979]). *Let G be a multigraph which has a good k-colouring of the edges. Then the edges of G admit a uniform good k-colouring.*

For if not, the edges of G admit a good k-colouring $(E_1,E_2,\ldots,E_k)$ with, for example, $|E_2| \ge |E_1| + 2$. The partial graph $G^{1,2}$ generated by the edges of colour 1 or 2 admits a (weak) bicolouring of the edges; thus $G^{1,2}$ has no connected component which is an odd cycle without chords. We show that from this we can find a bicolouring of the edges of $G^{1,2}$ which is uniform. We may suppose $G^{1,2}$ to be connected.

If all the vertices are of even degree, $G^{1,2}$ admits an Eulerian cycle, in which we can colour the edges alternately with the two colours. If the Eulerian cycle is odd we take as starting point a vertex of $G^{1,2}$ having degree greater than two (which is always possible since $G^{1,2}$ is not an odd cycle without chords). Thus the edges of $G^{1,2}$ admit a uniform bicolouring.

If there exist vertices of odd degree, G has $2p$ vertices of odd degree and there exists a partition of the edges into p paths joining the odd vertices in twos. An alternating colouring in two colours of the edges of each of these paths gives a uniform good 2-colouring. Thus we obtain a new colouring satisfying (1), and we can therefore apply Theorem 3.

Note that Corollary 2 is more general than Corollary 1, and that the values of k which guarantee a good k-colouring of G have been obtained by Fournier [1973]. (See also de Werra [1977]).

Given a hypergraph H, we call a "positional game on H" the situation where two players, say A and B, play in turn at colouring a vertex of H, with the colour red for A and the colour blue for B. A vertex already coloured cannot be recoloured; the winner is the one who first colours an edge of H completely with his colour. If neither of the players obtains a monochromatic edge then the game is a draw.

Example 1. Tic-Tac-Toe in p dimensions. This is played on the set of cells of a hypercube of p dimensions of sides equal to r, considered as a hypergraph on r^p vertices (the cells of the hypercube) in which the edges are all the sets of r cells that are in line. This game has been studied by Hales and Jewett [1963], who showed that if r is odd and $\geq 3^p-1$ or r is even and $\geq 2^{p+1}-2$, then player B can force a draw.

One can also play by trying to colour three points in a line with the same colour on any configuration at all, for example the projective plane with seven points.

Example 2. Ramsey games. Two players A and B play alternately colouring respectively in red and blue an edge of the complete graph K_n on n vertices; the first player to colour with his colour all the edges of a k-clique has won, and his opponent has lost. The hypergraph H_n which must be considered has $\binom{n}{2}$ vertices and is $\binom{k}{2}$-uniform. A celebrated theory of Ramsey states that there exists an integer $R(k,k)$ such that for every $n \geq R(k,k)$, the hypergraph H_n has no bicolouring (so that, in consequence, the first player has a winning strategy); if $n(k)$ denotes the smallest order for which the first player wins, we have $n(k) \leq R(k,k)$.

Fundamental Proposition. *In a positional game on a hypergraph H which admits no uniform bicolouring, the first player A has a strategy which assures him a win.*

Proof. If H does not have a uniform bicolouring, there necessarily exists a monochromatic edge when all the vertices have been coloured. Thus it is not possible to have a drawn game. This implies, by the theorem of Zermelo-von Neumann, that either player A or player B has a winning strategy.

We argue by contradiction, and suppose that it is the second player B who has a winning strategy σ. Thus, with the following sequence of moves:

$$x_1,\ y_1 = \sigma(x_1),\ x_2,\ \ y_2 = \sigma(x_1,x_2),\ x_3,\ \ y_3 = \sigma(x_1,x_2,x_3), etc.$$

the first monochromatic edge will be blue, B's colour. However the first player A can play according to the following rule: x_0 being an arbitrary vertex, A's first choice will be $x_1 = \sigma(x_0)$; A's second choice will be $x_2 = \sigma(x_0,y_1)$; etc. (If at any step, $y_i = x_0$, that is to say player B chooses the arbitrary vertex x_0, the player A will play in the same manner with $x_{i+1} = \sigma(x_0,y_1,y_2,\ldots,y_i')$, where y_i' is a new arbitrary vertex not already coloured). In this manner A is assured of obtaining a win, and the first monochromatic edge will be red: a contradiction.

Theorem 4. *Let H be a hypergraph such that*

$$(1)\qquad \sum_{E\in H} 2^{-|E|} + \max_x \sum_{E\in H(x)} 2^{-|E|} < 1.$$

Then H admits a uniform bicolouring. Furthermore in the positional game on H the second player B has a strategy ensuring a draw.

Proof. For a start, consider a hypergraph H satisfying (1), and let player A, who is trying to obtain a win, choose a vertex x_1. After this choice, player B must consider the hypergraph $H_1 = H_{X-\{x_1\}}$ to choose a vertex y_1. After this choice, player A must consider the partial hypergraph $H_1' = H_1 - H_1(y_1)$ to choose a vertex x_2, etc. This defines a sequence of hypergraphs H, H_1, H_1', H_2, H_2',.... It is then a matter of showing that B will never leave a hypergraph H_{i-1}' with a loop, or, equivalently, that A will never obtain a family of sets H_i having as "edge" the empty set. For simplicity let us set

$$v(H) = \sum_{E\in H} 2^{-|E|}.$$

Then the hypergraph $H_1 = H_{X-\{x_1\}}$ satisfies

$$v(H_1) = \sum_{E \in H(x_1)} 2^{-(|E|-1)} + \sum_{E \in H - H(x_1)} 2^{-|E|}$$

Thus, from (1),

(2) $$v(H_1) = v(H) + v[H(x_1)] < 1.$$

Let y_1 be the reply of player B; then the new hypergraph $H_1' = H_1 - H_1(y_1)$ to be considered satisfies

(3) $$v(H_1') = v(H_1) - v(H_1(y_1))$$

If B chooses a vertex y_1 which maximizes $v(H_1(y))$ then, whatever the choice x_2 of his opponent,

(4) $$v[H_1(x_2)] \leq v[H_1(y_1)].$$

After the choice x_2 of A, the new hypergraph $H_2 = [H_1']_{X-\{x_2\}}$ satisfies

$$v(H_2) = v(H_1') + v[H_1'(x_2)] \leq v_1(H_1') + v[H_1(x_2)]$$
$$= v[H_1] - v[H_1(y_1)] + v[H_1(x_2)] \leq v(H_1) < 1$$

by virtue of (2), (3) and (4).

If B plays in this manner on every occasion, we always have $v(H_i) \leq v(H_1) < 1$. The family H_i cannot have the empty set as an edge, since that would imply

$$v(H_i) \geq \frac{1}{2^0} = 1.$$

Thus B can force a draw, and consequently, from the fundamental proposition, H admits a uniform bicolouring.

Corollary (Erdős, Selfridge [1973]). *Let $H = (E_i/i \in I)$ be a hypergraph without loops, of anti-rank $s = \min_i |E_i|$, and such that the number of edges m and the maximum degree Δ satisfy $m + \Delta < 2^s$. Then H admits a uniform bicolouring. Furthermore, in a positional game on H, the second player B has a strategy for forcing a draw.*

Indeed, in this case we have

$$\sum_{E\in H} 2^{-|E|} + \max_x \sum_{E\in H(x)} 2^{-|E|} \leq m.2^{-s} + \Delta.2^{-s} < 1.$$

Theorem 5. *Let H be a hypergraph without loops, of order n such that*

$$\sum_{E\in H} \binom{n-|E|}{[n/2]} < \binom{n-1}{[n/2]}$$

Then H admits a uniform bicolouring.

Proof. Let $p = [n/2]$, and let $\mathcal{T}_p$ be the family of transversals of H having cardinality p. Consider the hypergraph

$$K_n^p - \mathcal{T}_p = \{F/F\subset X, |F|=p, F\cap E=\emptyset \text{ for some } E\in H\}.$$

We have

$$m(K_n^p) - m(\mathcal{T}_p) = m(K_n^p - \mathcal{T}_p) \leq \sum_{E\in H} \binom{n-|E|}{p} < \binom{n-1}{p}$$

therefore

$$m(\mathcal{T}_p) > \binom{n}{p} - \binom{n-1}{p} = \binom{n-1}{p-1}.$$

From the theorem of Erdős, Chao-Ko, Rado (Theorem 5, Chapter 1), this implies that $\mathcal{T}_p$ is not an intersecting family, and therefore contains two disjoint sets A and B. If n is even, (A,B) is a bicolouring of H which is uniform. If n is odd, we obtain such a bicolouring by adjoining to A the unique vertex of $X-(A\cup B)$.

Generalization (Hansen, Loréa [1978]). *Let H be a hypergraph of order $n \geq k$, and let $p = [\frac{n}{k}]$, $q = n-pk$. If*

$$k\sum_{E\in H} \binom{n-|E|}{n-p} + q\sum_{E\in H} \frac{|E|}{p+1-|E|} < \binom{n}{p}$$

then H admits a uniform k-colouring.

4. Extremal problems related to the chromatic number

Numerous works (mostly Hungarian) have as their object the study of the smallest number of edges (or the largest number of edges) which an r-uniform hypergraph of order n can have if some given property holds; these are often referred to collectively as "extremal problems". In most papers these results are obtained by "probabilistic methods" (cf. Erdős, Spencer [1974]); here we shall obtain the principal results as simple corollaries of theorems in chapter 3.

First let us consider the largest number of edges in an r-uniform hypergraph of order $\leq n$ which is k-colourable, that we denote by

$$M_k(n,r) = \max_{\substack{\chi(H)\leq k \\ n(H)\leq n}} m(H).$$

Let us consider also the smallest number of edges in an r-uniform hypergraph of order $\leq n$ which is not k-colourable, that we denote by

$$m_k(n,r) = \min_{\substack{\chi(H)>k \\ n(H)\leq n}} m(H)$$

Denote by $M_k^0(n,r)$ the largest value of m for which there exists an r-uniform hypergraph H with $n(H) \leq n$, $m(H) = m$, and such that by adding a set of $n-n(H)$ isolated points we can find a uniform k-colouring; denote by $m_k^0(n,r)$ the smallest number of edges in an r-uniform hypergraph of order $\leq n$ which has no uniform k-colouring (if we complete its set of vertices by adding isolated vertices up to a total of n). We then have

$$1 \leq m_k(n,r) \leq M_k(n,r) \leq \binom{n}{r}$$

$$1 \leq m_k^0(n,r) \leq M_k^0(n,r) \leq \binom{n}{r}$$

$$m_k^0(n,r) \leq m_k(n,r)$$

$$M_k^0(n,r) \geq M_k(n,r).$$

It is easy to calculate $M_k(n,r)$ and $M_k^0(n,r)$, which are given by the following result:

Theorem 6 (Sterboul [1974]). *Let $H_{n,k}^r$ be an r-uniform hypergraph of order n on X defined by a uniform k-partition $(Y_1,Y_2,\ldots,Y_k)$ of X and by*

$$H^r_{n,k} = (E/E \subset X;\ |E| = r,\ E \not\subset Y_1,\ E \not\subset Y_2, \ldots, E \not\subset Y_k)$$

Then we have

$$M_k(n,r) = M_k^0(n,r) = m(H^r_{n,k})$$

Moreover, every r-uniform k-colourable hypergraph of order n with $M_k(n,r)$ edges is isomorphic to $H^r_{n,k}$.

Proof. Clearly every r-uniform hypergraph of order n having a uniform k-colouring contains $H^r_{n,k}$ as a partial hypergraph. Furthermore, if H is an r-uniform hypergraph of order n with $\chi(H) \leq k$, consider a k-colouring $(S_1,S_2,\ldots,S_k)$ of H; let $|S_i| = n_i$. We have

$$m(H) \leq \binom{n}{r} - \sum_{i=1}^{k} \binom{n_i}{r} \leq \binom{n}{r} - \min_{\Sigma n_i = n} \sum_{i=1}^{k} \binom{n_i}{r}.$$

It is easy to see that the minimum of $\sum_{i=1}^{k} \binom{n_i}{r}$ for $n_1 + n_2 + \cdots + n_k = n$ is obtained if and only if we have

$$[\frac{n}{k}] \leq n_i \leq [\frac{n}{k}]^* \qquad (i = 1,2,\ldots,k).$$

Indeed, we verify that $n_1 \geq n_2 + 2$ implies

$$\binom{n_1}{r} + \binom{n_2}{r} > \binom{n_1-1}{r} + \binom{n_2+1}{r}.$$

This algebraic lemma shows that

$$m(H) \leq m(H^r_{n,k}).$$

This shows also that equality holds only if the k-colouring $(S_1,S_2,\ldots,S_k)$ is uniform. The result follows.

It is more difficult to calculate $m_k(n,r)$. We have $m_2(n,2) = 3$ for $n \geq 3$ (since the triangle K_3 is not bicolourable); $m_2(5,3) \leq 10$ (since K_5^3 is not bicolourable); $m_2(n,3) = 7$ for $n \geq 7$ (since P_7 is not bicolourable). In the case of graphs we easily find that $m_k(n,2) = \binom{k+1}{2}$ for $n \geq k+1$, and the only extremal graph is K_{k+1} (cf. *Graphs*, Theorem 4, Chapter 15).

Theorem 7 (Erdős [1963]). *For $r \geq 2$, $k \geq 2$, $n \geq kr$, we have*

$$m_k(n,r) \geq k^{r-1}.$$

Proof.

1. Let X be a set of cardinality n, and let $\pi = (S_1, S_2, \ldots, S_k)$ be an ordered *k-partition* of X, that is to say, a sequence of k disjoint subsets whose union is X, (some of which could be empty). Consider the hypergraph $H_0 = (E_\pi / \pi)$ whose vertices are the r-tuples of X, an edge E_π being the set of r-tuples completely contained in a single class of the partition π.

Every set of edges of an r-uniform hypergraph on X with no k-colouring defines a transversal of H_0, and vice versa; hence

$$m_k(n,r) = \tau(H_0)$$

We have $m(H_0) = k^n$, for we can identify an ordered k-partition with a sequence of n integers taken from $\{1,2,\ldots,k\}$. Moreover, we have $\Delta(H_0) = k^{n-r} \times k$.

From Theorem 1, chapter 3, we have then

$$m_k(n,r) = \tau(H_0) \geq \frac{m(H_0)}{\Delta(H_0)} = k^{r-1}.$$

Q.E.D.

Remark. By some more or less complicated algebraic manipulations, we can improve the lower bound in Theorem 7, using the inequality $\tau(H_0) \geq \tau^*(H_0)$.

For $k = 2$ the best lower bound for $m_k(n,r)$ has been obtained by Beck [1977], [1978]: for every $\epsilon > 0$ and every $n \geq n(\epsilon)$ we have $m_2(n,r) \geq 2^r r^{\frac{1}{3}-\epsilon}$. The inequality $m_2(n,r) \leq 2^r r^2$, due to Erdős [1964] and Schmidt [1964], has also been improved by Seymour [1974], giving, for example, $m_2(n,4) \leq 23$, $m_2(n,5) \leq 51$.

Generalisation (Hansen, Loréa [1978]). *Let H be a hypergraph of order n such that*

$$\sum_{E \in H} \frac{k^{-|E|}(k^2-k+1)}{|E|} \cdot \sum_{E \in H} k^{-|E|} < 1.$$

Then $\chi(H) \leq k$.

(The proof is analogous to that of Theorem 5).

Corollary 1 (Johnson [1976]). *For $r \geq 2$, $k \geq 2$, $n \geq kr$ we have*

$$m_k(n,r) \geq \frac{rk^r}{r+k(k-1)}.$$

Corollary 2 (Schmidt [1964], Herzog, Schönheim [1972]). *For $k \geq 2$, $n \geq 2r$, we have*

$$m_2(n,r) \geq \frac{r.2^r}{r+2}.$$

For $k = 2$ we have upper bounds due to Erdős [1964], Chvátal [1971], Beck [1977], Erdős and Spencer [1974]. Some bounds with the maximum degree (in place of the number of edges) are due to Erdős and Lovász [1975].

We propose now to find some bounds for $m_k^0(n,r)$.

Theorem 8. *Let $r \geq 2$, $k \geq 2$, $n \geq kr$. In a uniform k-partition of X with $|X| = n$, let q_1 be the number of classes of size $[\frac{n}{k}]$, and let q_2 be the number of size $[\frac{n}{k}]^*$. We have*

$$m_k^0(n,r) \geq \binom{n}{r}\left[q_1\binom{[n/k]}{r} + q_2\binom{[n/k]^*}{r}\right]^{-1}$$

Proof. Define (as in the proof of Theorem 7) a hypergraph $H_0 = (E_\pi)$ whose vertices are all the r-tuples of X; for every uniform k-partition π, E_π denotes the set of r-tuples contained in a single class of the partition. Clearly H_0 is regular, and it is also uniform of rank

$$r(H_0) = q_1\binom{[n/k]}{r} + q_2\binom{[n/k]^*}{r}.$$

Thus, using Theorem 1 of Chapter 3, we obtain

$$m_k^0(n,r) = \tau(H_0) \geq \frac{n(H_0)}{r(H_0)} = \binom{n}{r}\left[q_1\binom{[n/k]}{r} + q_2\binom{[n/k]^*}{r}\right]^{-1}$$

Remark. The value of $m_k^0(n,r)$ is precisely known when $r = 2$. We give first some examples of graphs of order n having no uniform k-colouring.

If $n \leq k$, every graph of order n has a uniform k-colouring.

If $n > k$, consider the graph $G_1(n,k)$ formed by the union of a clique K_{k+1} (with $k+1$ vertices) and a stable set S_{n-k-1} (with $n-k-1$ vertices). This graph is certainly of order n, and having no k-colouring, it has no uniform k-colouring.

If $k < n \leq 2k$, consider the graph $G_2(n,k)$ formed by the union of a clique K_{2k-n+1} and a stable set $S_{2n-2k-1}$, together with all the edges joining one to the other. This graph certainly has n vertices, and it will be left as an exercise to the reader to verify that it has no uniform k-colouring.

If $n \geq 2k$, consider the graph $G_3(n,k)$ formed from the union of a set A of cardinality 1, a set B of cardinality $n-[\frac{n}{k}]+1$, a set C of cardinality $[\frac{n}{k}]-2$, and all the edges joining the singleton of A to the elements of B. This graph is certainly of order n, and the task of verifying that it has no uniform k-colouring is left to the reader, by way of an exercise.

Thus, for every $n > k$, the minimum number of edges in a graph of order n with no uniform k-colouring satisfies

$$(1) \qquad m_k^0(n,2) \leq \min_i m[G_i(n,k)].$$

Indeed, Berge and Sterboul [1977] showed that equality holds in (1). Further, they determined the structure of all graphs of order n with no uniform k-colouring having $m_k^0(n,2)$ edges.

The same extremal problems can be formulated for the stability number.

Proposition. *Let n,p,r be integers such that $n \geq p \geq r \geq 2$. The maximum number of edges in an r-uniform hypergraph of order n having a stable set of cardinality p is*

$$\max_{\alpha(H)\geq p} m(H) = \binom{n}{r} - \binom{p}{r}.$$

Indeed, the only extremal hypergraph is an r-uniform hypergraph H_0 on X with $|X| = n$, defined by considering a set $S \subset X$ with $|S| = p$, and setting:

$$H_0 = (E / E \subset X,\ |E| = r,\ E \cap (X-S) \neq \emptyset).$$

Clearly, we have

$$m(H_0) = \binom{n}{r} - \binom{p}{r}$$

as was to be proved.

For $n \geq p \geq r \geq 2$, the *Turan number* $T(n,p,r)$ is the smallest number of edges in an r-uniform hypergraph of order n such that every set of vertices of cardinality p contains at least one edge. That is to say,

$$T(n,p,r) = \min_{\alpha(H)<p} m(H).$$

Example 1 (Turan [1941]). Consider a set X with $|X| = n$, and a uniform $(p-1)$-partition $(S_1,S_2,\ldots,S_{p-1})$ of X. The graph $G_{n,p-1}$ obtained by joining two elements (vertices) of X if and only if they belong to the same S_i satisfies $\alpha(G_{n,p-1}) < p$. Turan showed that it is the only graph with this property having the minimum number of edges. Thus

$$T(n,p,2) = m(G_{n,p-1}).$$

(cf. *Graphs*, Theorem 5, Chapter 13).

Example 2. Consider the 3-uniform hypergraph on $X = \{1,2,\ldots,9\}$ whose edges are: 123, 456, 789, 147, 258, 369, 159, 267, 348, 168, 249, 357 (the "affine plane of rank 3"). It can be shown that this is the only extremal 3-uniform hypergraph with $\alpha < 5$. Thus $T(9,5,3) = 12$.

Few values of $T(n,p,r)$ are known, but it is known that when $n \to \infty$ the function $T(n,p,r)\binom{n}{r}^{-1}$ tends to a limit $t(p,r)$ (Katona, Nemetz, Simonovits [1964]). For $p > r \geq 3$ no values of $t(p,r)$ are known, but it is known that $t(p,r) \geq \binom{p-1}{r-1}^{-1}$ (de Caen [1983]). The best upper bound for $t(r+1,r)$ is due to Frankl and Rödl [1985].

Theorem 9. *For $n \geq p \geq r \geq 2$ we have*

(1) $$T(n,p,r) \geq \binom{n}{r}\binom{p}{r}^{-1}.$$

(2) $$T(n,p,r) \leq \left[1+\log\binom{n-r}{p-r}\right]\binom{n}{r}\binom{p}{r}^{-1}.$$

Proof. Let X be a set with $|X| = n$.

Let H_0 be the hypergraph whose vertices are the r-tuples of X, and for $S \subset X$ with $|S| = p$, the edge E_S denotes the set of r-tuples of X contained in S. Then $T(n,p,r) = \tau(H_0)$. Furthermore

$$n(H_0) = \binom{n}{r}, \quad r(H_0) = \binom{p}{r}, \quad m(H_0) = \binom{n}{p}, \quad \Delta(H_0) = \binom{n-r}{p-r}$$

From Theorem 1 of Chapter 3, we have

$$T(n,p,r) = \tau(H_0) \geq \frac{m(H_0)}{\Delta(H_0)} = \frac{n(H_0)}{r(H_0)} = \binom{n}{r}\binom{p}{r}^{-1},$$

from which (1) follows.

Theorem 12 of Chapter 3 gives

$$T(n,p,r) = \tau(H_0) \leq \left[1+\log \Delta(H_0)\right]\tau^*(H_0)$$
$$= \left[1+\log\binom{n-r}{p-r}\right]\binom{n}{r}\binom{p}{r}^{-1}$$

from which (2) follows.

Remark. The inequality (1) was originally found (by different methods) by Katona, Nemetz, Simonovits [1964]. By generalizing a theorem of Moon and Moser, de Caen [1983] has been able to improve (1) to

(3) $$T(n,p,r) \geq \frac{n-p+1}{r}\binom{n}{r-1}\binom{p-1}{r-1}^{-1}.$$

(For a more complete account of Turan numbers the reader should refer to Brouwer, Voorhoeve [1978]).

Note that (2) improves a bound due to Schönheim [1964].

Corollary. *Let H be an r-uniform hypergraph of order n with m edges; then $\alpha(H) \geq nm^{-1/r}$.*

Indeed, if for an integer p we have $m \leq (np^{-1})^r$, then $m < \binom{n}{r}\binom{p}{r}^{-1}$, and from (1), $m < T(n,p,r)$. In other words $p \leq nm^{-1/r}$ implies $\alpha(H) \geq p$, whence

$$\alpha(H) \geq nm^{-1/r}.$$

5. Good edge-colourings of a complete hypergraph

Let k be an integer ≥ 2. A *weak k-colouring of the edges* of a hypergraph H is the colouring defined by a weak k-colouring of the dual hypergraph H^*. It is thus a partition $H = H_1+H_2+...+H_k$ (edge-disjoint sum) such that for every vertex x with $d_H(x) > 1$, the star $H(x)$ has at least two edges of different colours. A *good k-colouring of the edges* of H is a weak k-colouring of the edges of H such that if $d_H(x) \geq k$, the star $H(x)$ contains at least one edge of each of the colours, and if $d_H(x) \leq k$, the edges of the star $H(x)$ all have different colours. A *strong k-colouring of the edges* of H is a partition $H = H_1+H_2+...+H_k$ such that the edges of the star $H(x)$ all have different colours. The *chromatic index* of H is the smallest value of k for which a strong k-colouring of the edges exists; it is thus the strong chromatic number $\gamma(H^*)$.

In this section we shall determine for what values of k the r-partite complete hypergraph and the r-complete hypergraph have a good k-colouring of the edges.

Theorem 10 (Meyer [1975]). *For every $k \geq 2$, the edges of the complete r-partite hypergraph admit a good k-colouring.*

(*) **Proof.** Let $H = K^r_{n_1,n_2,\dots,n_r}$, with $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$, and let $X^i = \{0,1,...,n_i-1\}$ denote the i-th class.

We have seen (Theorem 9, Chapter 1) that for $p = \prod_{i \neq 1} n_i = \Delta(H)$, we obtain a strong p-colouring of the edges by allocating to the edge $\bar{x} = x^1x^2 \cdots x^r$ the $(r-1)$-tuple $(\alpha_2,\alpha_3, \dots ,\alpha_r)$, where

$$\alpha_i = [x^i+x^1]_{n_i}.$$

Thus there exists a good k-colouring for every $k \geq p$; for if $k > p$ it suffices to complete the p-colouring above with $k-p$ empty classes.

We can also verify that for $n_q \leq s \leq n_{q+1}$ and for $p = s \prod_{\substack{i \neq q \\ i \neq q+1}} n_i$, we obtain a good p-colouring by allocating to the edge $\bar{x}$ the $(r-1)$-tuple $(\alpha_2,\alpha_3, \dots ,\alpha_r)$, where

$$\begin{aligned}\alpha_i &= [x^{i-1}+x^i]_{n_i-1} \text{ if } 2 \le i \le q\\ &= [x^q+x^{q+1}]_s \text{ if } i = q+1\\ &= [x^{i-1}+x^i]_{n_i} \text{ if } q+1 < i \le r.\end{aligned}$$

For $k \le \prod_{i \ne r} n_i = \min d_H(x)$, we obtain a good k-colouring $(S_1,S_2,\ldots,S_{k-1},\bigcup_{i=k}^{p} S_i)$ from the p-colouring $(S_1,S_2,\ldots,S_p)$ defined by the formula above with $q+1 = r$ and $s = n_{r-1}$. For all the other values of k we find a $(\alpha_1,\alpha_2, \ldots ,\alpha_r)$ by analogous formulae (we refer the reader to Mayer [1975]).

We note also (without proof):

Generalization (Baranyai [1978]). *For every $k \ge 2$ the edges of the complete r-partite hypergraph admit an equitable, k-colouring which is uniform.*

The existence of good k-colourings of the edges of the hypergraph K_n^r has been proved by Baranyai by induction on the order n. In order that the inductive method can be used, it is necessary to aim for a stronger statement than we are going to prove. First we say that a hypergraph H on X is *almost-regular* if we have

$$|d_H(x)-d_H(y)| \le 1 \qquad (x,y \in X).$$

Lemma 1. *Let H be a hypergraph on X. If, for a vertex $a \in X$, the subhypergraph H' induced by $X - \{a\}$ is almost-regular, and if*

$$\left[\frac{1}{n} \sum_{E\in H} |E|\right] \le d_H(a) \le \left[\frac{1}{n} \sum_{E\in H} |E|\right]^*,$$

then H is almost-regular.

(*) Set $\alpha = \sum_{E\in H} |E|$, so $[\frac{\alpha}{n}] \le d_H(a) \le [\frac{\alpha}{n}]^*$. For $x \ne a$ we can show that

$$\left[\frac{\alpha-d_H(a)}{n-1}\right] \le d_H(x) = d_{H'}(x) \le \left[\frac{\alpha-d_H(a)}{n-1}\right]^*.$$

If we note that

$$[\frac{\alpha}{n}] = \left[\frac{\alpha - [\frac{\alpha}{n}]^*}{n-1}\right], \quad [\frac{\alpha}{n}]^* = \left[\frac{\alpha - [\frac{\alpha}{n}]}{n-1}\right]^*$$

we deduce that

$$[\frac{\alpha}{n}] \leq d_H(x) \leq [\frac{\alpha}{n}]^* \qquad (x \neq a).$$

This shows that H is almost-regular.

Lemma 2. *Let ϵ^i_j, for $i = 1,2,...,s$, $j = 1,2,...,t$, be real numbers ≥ 0. There exist integers $e^i_j \geq 0$ such that*

(i) $\qquad [\epsilon^i_j] \leq e^i_j \leq [\epsilon^i_j]^*$

(ii) $\qquad \left[\sum_i \epsilon^i_j\right] \leq \sum_i e^i_j \leq \left[\sum_i \epsilon^i_j\right]^*$

(iii) $\qquad \left[\sum_j \epsilon^i_j\right] \leq \sum_j e^i_j \leq \left[\sum_j \epsilon^i_j\right]^*$

(*) **Proof.** Consider a transport network R whose vertices consist of a source a, a sink z, and two sets S and T. The arcs of R, each with an upper and lower capacity, are of three kinds.

1) arcs (a,i), $i \in S$, able to bear a flow ψ with

$$\left[\sum_j \epsilon^i_j\right] \leq \psi(a,i) \leq \left[\sum_j \epsilon^i_j\right]^*$$

2) arcs (i,j), $i \in S$, $j \in T$, able to bear a flow ψ with

$$[\epsilon^i_j] \leq \psi(i,j) \leq [\epsilon^i_j]^*$$

3) arcs (j,z), $j \in T$, able to bear a flow ψ with

$$\left[\sum_j \epsilon_j^i\right] \leq \psi(j,z) \leq \left[\sum_i \epsilon_j^i\right]^*.$$

The necessary and sufficient conditions for the existence of a flow in a network with integer capacities are the same for a flow with real values and for a flow with integer values (cf. *Graphs*, Chapter 5, §2). Since the network R admits a real-valued flow $\overline{\psi}$ with $\overline{\psi}(i,j) = \epsilon_j^i$, it will admit an integer-valued flow ψ. The integers $\psi(i,j) = e_j^i$ satisfy the conditions (i), (ii) and (iii).

Baranyai's Lemma. *Let n, r_i and m_j^i, for $i \in I$, $j \in J$, be integers satisfying*

(I) $\quad 0 \leq r_i \leq n \qquad (i \in I);$

(II) $\quad m_j^i \geq 0 \qquad (i \in I,\ j \in J);$

(III) $\quad \sum_{j \in J} m_j^i = \binom{n}{r_i} \qquad (i \in I).$

Then there exists a set X with $|X| = n$ and families $H_j^i = (E_j^i(\lambda))$ of subsets of X satisfying

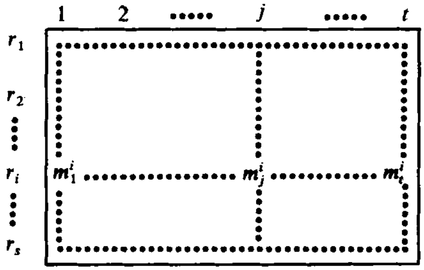
(1) $\quad m(H_j^i) = m_j^i;$

(2) $\quad H^i = \sum_{j \in J} H_j^i$ *is the complete hypergraph* $K_n^{r_i}$;

(3) $\quad H_j = \sum_{i \in I} H_j^i$ *is almost-regular, or, equivalently,*

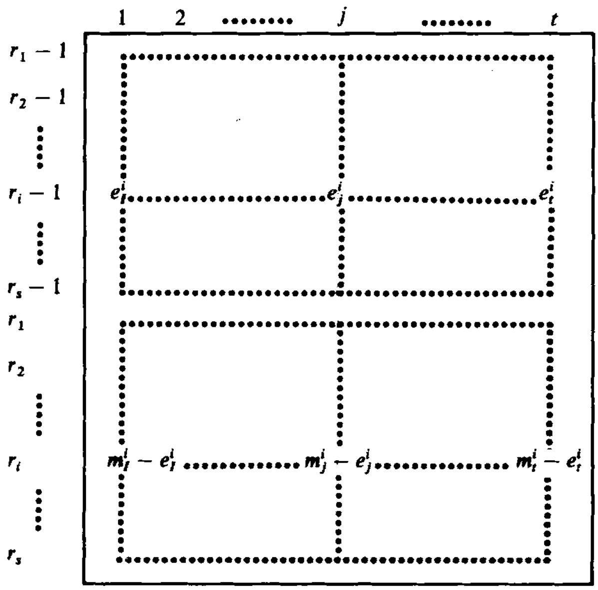
$$\left[\frac{1}{n} \sum_{E \in H_j} |E|\right] \leq d_{H_j}(x) \leq \left[\frac{1}{n} \sum_{E \in H_j} |E|\right]^* \qquad (x \in X).$$

(*) Proof. We shall suppose that the assertion is verified for every integer $< n$, and prove it to be true for n. Consider, for n, the following tableau of integers satisfying (I), (II) and (III).



We can eliminate from this matrix every row i_0 with $r_{i_0} = 0$ (since $\sum_j m_j^{i_0} = 0$ from (III), and $H^{i_0} = \emptyset$). Similarly we can eliminate every row i_1 with $r_{i_1} = n$ (since $H^{i_1} = (X)$, from (III), and its suppression will make no change in the conclusion).

Supposing this to have been done, consider, with $n-1$, the new tableau:



where the e_j^i are integers satisfying properties (i), (ii) and (iii) of lemma 2 with $\epsilon_j^i = \dfrac{1}{n} r_i m_j^i$. Then the coefficients in the new tableau satisfy

(I′) $\qquad 0 \leq r_i - 1 \leq n-1$

(I″) $\qquad 0 \leq r_i \leq n-1$

(II′) $\qquad e_j^i \geq 0$

(II″) $\qquad m_j^i - e_j^i \geq 0$

In order to obtain (II″), observe that

$$m_j^i - e_j^i \geq \frac{r_i m_j^i}{n} - e_j^i.$$

The first term being an integer we deduce, by using (i) of Lemma 2, that

$$m_j^i - e_j^i \geq \left[\frac{r_i m_j^i}{n}\right]^* - e_j^i \geq 0.$$

We also have, from (iii) and (III),

$$\sum_j e_j^i \geq \left[\sum_j \frac{r_i m_j^i}{n}\right] = \left[\binom{n}{r_i}\frac{r_i}{n}\right] = \binom{n-1}{r_i-1}.$$

For the same reason we have the inverse inequality, thus

(III′) $$\sum_j e_j^i = \binom{n-1}{r_i-1}.$$

Finally

(III″) $$\sum_j \left[m_j^i - e_j^i\right] = \binom{n}{r_i} - \binom{n-1}{r_i-1} = \binom{n-1}{r_i}$$

By virtue of the induction hypothesis, there exists a set $\overline{X}$ with $|\overline{X}| = n-1$, and families $\overline{H}_j^i = (\overline{E}_j^i(\lambda))$ and $\overline{\overline{H}}_j^i = (\overline{\overline{E}}_j^i(\lambda))$ of subsets of $\overline{X}$ satisfying

(1′) $\qquad m(\overline{H}_j^i) = e_j^i$

(1″) $\quad m(\overset{\Rightarrow}{H}_j) = m_j^i - e_j^i$

(2′) $\quad \sum_j \overline{H}_j^i = K_{n-1}^{r_i-1}$

(3′) $\quad \sum_j \overset{\Rightarrow}{H}_j = K_{n-1}^{r_i-1}$

(4) $\quad \sum_i \overline{H}_j^i + \sum_i \overset{\Rightarrow}{H}_j$ is almost-regular.

Consider an additional point a. Set $X = \overline{X} \cup \{a\}$ and

$$E_j^i(\lambda) = \overline{E}_j^i(\lambda) \cup \{a\} \text{ for } 1 \leq \lambda \leq e_j^i$$
$$= \overset{\Rightarrow}{E}_j(\lambda) \qquad \text{for } e_j^i{+}1 \leq \lambda \leq m_j^i.$$

It is clear that the hypergraphs $H_j^i = (E_j^i(\lambda)/1 \leq \lambda \leq m_j^i)$ satisfy

$$\sum_{j \in J} H_j^i = K_n^{r_i}.$$

Furthermore

$$\sum_{i,\lambda} \frac{|E_j^i(\lambda)|}{n} = \sum_i \frac{r_i m_j^i}{n}.$$

Thus, for $x \neq a$,

$$\left[\frac{1}{n}\sum_{i,\lambda} |E_j^i(\lambda)|\right] \leq d_{H_j}(x) \leq \left[\frac{1}{n}\sum_{i,\lambda} |E_j^i(\lambda)|\right]^*.$$

From Lemma 1, we see then that $H_j = \sum_i H_j^i$ is amost-regular, which completes the proof.

Theorem 11 (Baranyai [1975]). *Let n, r be integers, $n \geq r \geq 2$, and let $m_1, m_2, \ldots, m_t$ be integers with $m_1+m_2+\cdots+m_t = \binom{n}{r}$. Then K_n^r is the edge-disjoint sum of t hypergraphs H_j, each satisfying*

(1) $\quad m(H_j) = m_j$

(2) $$\left[\frac{rm_j}{n}\right] \leq d_{H_j}(x) \leq \left[\frac{rm_j}{n}\right]^* \qquad (x \in X).$$

This is the statement of Baranyai's lemma for $|I| = 1$.

Corollary 1 (Baranyai). *K_n^r is the edge-disjoint sum of partial h-regular hypergraphs H_j if and only if r divides hn and $\frac{hn}{r}$ divides $\binom{n}{r}$. In this case, the H_j make up a uniform colouring of the edges of K_n^r.*

Proof. If there exists a decomposition of K_n^r as the sum of h-regular hypergraphs H_j, we have $rm(H_j) = hn$ (by counting, in two different ways, the edges of the vertex-edge incidence graph). Thus r divides hn, and $\frac{hn}{r} = m(H_j)$ divides $m(K_n^r) = \binom{n}{r}$.

Conversely, if these conditions are satisfied, apply Theorem 11 with $m_j = \frac{hn}{r}$ and $t = \binom{n}{r}\frac{r}{hn}$. There exists a decomposition of K_n^r into t hypergraphs H_j such that

$$h = \left[\frac{rm_j}{n}\right] \leq d_{H_j}(x) \leq \left[\frac{rm_j}{n}\right]^* = h.$$

Thus the hypergraphs H_j are h-regular.

Corollary 2. *The complete graph K_n is the sum of h-regular graphs if and only if hn is even, and $\frac{hn}{2}$ divides $\binom{n}{2}$.*

Corollary 3 (Baranyai). *The hypergraph K_n^r has the coloured edge property if and only if r divides n. In this case, there exists an optimal colouring of the edges which is uniform.*

Proof. We note that $\frac{n}{r}$ divides $\binom{n}{r}$, the quotient being $\binom{n-1}{r-1}$. We therefore apply Corollary 1 with $h = 1$.

Corollary 4 (Baranyai). *The chromatic index of K_n^r is*

$$q(K_n^r) = \left[\binom{n}{r}\left[\frac{n}{r}\right]^{-1}\right]^*.$$

Proof. Let $K_n^r = H_1+H_2+...+H_q$ be a decomposition of the edges of K_n^r into $q = q(K_n^r)$ matchings. We have

$$\binom{n}{r} = m(K_n^r) = |H_1|+|H_2|+...+|H_q| \leq q\left[\frac{n}{r}\right].$$

Thus

$$q(K_n^r) \geq \left[\binom{n}{r}\left[\frac{n}{r}\right]^{-1}\right]^*.$$

On the other hand, if we denote by t the second term of this inequality, we can apply Theorem 11 with

$$m_1 = m_2 = \cdots = m_{t-1} = \left[\frac{n}{r}\right]$$

$$m_t = \binom{n}{r} - (t-1)\left[\frac{n}{r}\right] \leq \left[\frac{n}{r}\right].$$

Thus there exists a decomposition $K_n^r = H_1 + H_2 + \cdots + H_t$ such that, for every $x \in X$,

$$0 \leq d_{H_i}(x) \leq \left[\frac{r}{n}\left[\frac{n}{r}\right]\right]^* \leq 1.$$

This is then a strong colouring of the edges of K_n^r in t colours, whence $q(K_n^r) \leq t$, which completes the proof.

Corollary 5. *Let $K_n^r = H_1+H_2+...+H_p$ be a decomposition of the edges of K_n^r into p hypergraphs on X (" coverings"). If $p(K_n^r)$ denotes the smallest integer p for which such a decomposition exists, then*

$$p(K_n^r) = \left[\binom{n}{r}\left[\frac{n}{r}\right]^{*-1}\right].$$

The proof is the same as that above.

Corollary 6. *There exists a good k-colouring of the edges of K_n^r if and only if either $k \leq \left[\binom{n}{r}\left[\frac{n}{r}\right]^{*-1}\right]$ or $k \geq \left[\binom{n}{r}\left[\frac{n}{r}\right]^{-1}\right]^*$.*

Proof. Using Corollaries 4 and 5 we can write

$$p(K_n^r) = \left[\binom{n}{r}/\left[\frac{n}{r}\right]^{*-1}\right] \leq \frac{\binom{n}{r}}{\left[\frac{n}{r}\right]^*} \leq \Delta(K_n^r) = \binom{n}{r}\frac{r}{n}$$

$$\leq \frac{\binom{n}{r}}{\frac{n}{r}} \leq \left[\binom{n}{r}/\left[\frac{n}{r}\right]^{-1}\right]^* = q(K_n^r).$$

If $k < q(K_n^r)$ there is no strong k-colouring, and if $k \geq p(K_n^r)$ there exists no decomposition into k coverings. Hence there are no good k-colourings.

On the other hand, if $k \geq q(K_n^r)$ there exists an obvious good k-colouring, obtained from a colouring in $q(K_n^r)$ colours by adding empty classes. If $k \leq p(K_n^r)$, there exists a decomposition into k coverings, obtained from a decomposition into $p(K_n^r)$ coverings by redistributing the $p(K_n^r)-k$ last classes.

6. An application to an extremal problem

The above results enable us to give a partial answer to the following problem: what is the largest number of edges in an r-uniform hypergraph of order $\leq n$ which does not have $k+1$ pairwise disjoint edges. This number will be denoted by

$$M_k^l(n,r) = \max_{\nu(H)\leq k} m(H).$$

For the case of graphs this problem has already been solved by Erdős and Gallai [1959] (cf. *Graphs*, Theorem 2, Chapter 7).

Theorem 12. *Let n,r,k be integers with $n \geq r \geq 2$, $n \geq kr$. If we let*

$$q = \left[\binom{n}{r}/\left[\frac{n}{r}\right]^{-1}\right]^*$$

we have:

$$M_k^l(n,r) \leq (q-1)k + \min\{k, \binom{n}{r}-(q-1)\left[\frac{n}{r}\right]\}.$$

Proof. Let H be an r-uniform hypergraph on X, $|X| = n$, having no matching with $k+1$ edges, and with the largest number of edges possible. As in Corollary 4 of Theorem 11, consider the decomposition of the r-complete hypergraph K_n^r on X as the sum of q matchings H_j with

$$m(H_j) = [\frac{n}{r}] \qquad (j = 1,2,...,q-1)$$

$$m(H_q) = \binom{n}{r} - (q-1)[\frac{n}{r}].$$

The hypergraph H admits at most k edges in H_j for $j \leq q-1$, and at most $\min\{k, \binom{n}{r}-(q-1)[\frac{n}{r}]\}$ edges in H_q. Thus $m(H) = M_k'(n,r)$ is bounded by the expression given above in the statement of the theorem, as was to be proved.

Remark. If $k = 1$ and $r < \frac{n}{2}$, we have $M_1'(n,r) = \binom{n-1}{r-1}$ from the theorem of Erdös, Chao-Ko and Rado, and the only extremal hypergraph is the star $K_n^r(x_0)$.

If $n \leq kr+(r-1)$, we have $M_k'(n,r) = \binom{n}{r}$; the only extremal hypergraph is K_n^r.

If $n \geq kr+r$, consider a set X with $|X| = n$, a set $Y \subset X$ with $|Y| = k$, and let

$$\mathcal{E}_{n,k}^r = (E/E \subset X,\ |E| = r,\ E \cap Y \neq \emptyset).$$

The hypergraph $\mathcal{E}_{n,k}^r$ cannot have $k+1$ disjoint edges (for each of these edges would have to meet a distinct point of Y). Erdös [1965] proved that for $n > c_r k$, where c_r is a constant depending only on r, $\mathcal{E}_{n,k}^r$ is an extremal hypergraph; that is to say

$$M_k'(n,r) = m(\mathcal{E}_{n,k}^r) = \binom{n}{r} - \binom{n-k}{r}.$$

Furthermore, Erdös conjectured that for every $n \geq kr+r$, one of the hypergraphs K_{kr+r-1}^r or $\mathcal{E}_{n,k}^r$ is extremal, and consequently

$$M_k'(n,r) = \max\{\binom{kr+r-1}{r}, \binom{n}{r} - \binom{n-k}{r}\}.$$

When n is sufficiently large, is $\mathcal{E}_{n,k}^r$ the only extremal hypergraph? Bollobás, Daykin and Erdös [1980] showed that every r-uniform hypergraph H satisfying

$$n(H) > 2r^3k$$

$$m(H) > m(\mathcal{E}_{n,k}^r) - \binom{n-k-r}{r-1}+1$$

$$\nu(H) \leq k$$

is contained in an $\mathcal{E}_{n,k}^r$.

7. Kneser's problem

The study of the chromatic index of a hypergraph is comparable to the dual problem: What is the smallest number of intersecting families whose union is the set of edges of the hypergraph H? This new coefficient, denoted $\tau_0(H)$, and sometimes called the *Kneser number*, has properties similar to those of the transversal number $\tau(H)$. We have $\tau_0(H) \leq \tau(H)$, for one can always cover the set of edges of H with $\tau(H)$ stars. If H satisfies the Helly property, we clearly have $\tau_0(H) = \tau(H)$.

The study of $\tau_0(H)$ is inseparable from that of $\Delta_0(H)$ (the maximum cardinality of an intersecting family) and $\rho_k(H)$ (the minimum number of intersecting families which, collectively, cover each edge of H at least k times). The coefficient

$$\tau_0^*(H) = \min_{k \geq 1} \frac{\rho_k(H)}{k}$$

is sometimes called the *fractional Kneser number.*

Theorem 13. *For every hypergraph H,*

$$\nu(H) \leq \max_{H' \subset H} \frac{m(H')}{\Delta_0(H')} \leq \tau_0^*(H) = \min_{k \geq 1} \frac{\rho_k(H)}{k} \leq \max_{k \geq 1} \frac{\rho_k(H)}{k} = \tau_0(H) \leq \tau(H).$$

Proof. To the hypergraph $H = (E_1, E_2, ..., E_m)$ on X let us make correspond a hypergraph $\overline{H} = (\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m)$ on the set of intersecting families of H, where $\overline{E}_i$ is the set of intersecting families which contain E_i. We then have $E_i \cap E_j = \emptyset$ if and only if $\overline{E}_i \cap \overline{E}_j = \emptyset$. Moreover,

$$\nu(\overline{H}) = \nu(H)$$

$$\Delta(\overline{H}) = \Delta_0(H)$$

$$\tau_k(\overline{H}) = \rho_k(H)$$

$$\tau(\overline{H}) = \tau_0(H)$$

$$\tau^*(\overline{H}) = \tau_0^*(H)$$

Applying Theorem 1 of Chapter 3 to the hypergraph $\overline{H}$, we obtain the stated inequalities.

Example 1. Let P_7 be the projective plane on 7 points. We have

$$\Delta_0(P_7) = 7$$

$$\tau_0(P_7) = 1$$

We have also $\tau_0^*(P_7) = 1$, since Theorem 13 gives

$$1 = \nu(P_7) \le \tau_0^*(P_7) \le \tau_0(P_7) = 1.$$

Example 2. Let K_n^r be the r-complete hypergraph with $r \le \frac{n}{2}$. From the theorem of Erdős, Chao-Ko and Rado,

$$\Delta_0(K_n^r) = \binom{n-1}{r-1}.$$

We have also $\tau_0^*(K_n^r) = \frac{n}{r}$, since Theorem 13 gives

$$\frac{m(K_n^r)}{\Delta_0(K_n^r)} = \frac{\binom{n}{r}}{\binom{n-1}{r-1}} = \frac{n}{r} \le \tau_0^*(K_n^r) \le \frac{\rho_r(K_n^r)}{r} \le \frac{n}{r}.$$

We note that $\rho_r(K_n^r) \le n$, being given that the n stars of K_n^r collectively cover every edge exactly r times.

The problem of determining $\tau_0(K_n^r)$ which was put by Kneser in 1955, was not solved until 23 years later, by Lovász, using algebraic topological methods. We shall give here a simpler proof due to Baranyai [1978].

Proposition. *Let H be an r-uniform hypergraph of order $n \ge 2r$. Then $\tau_0(H) \le n-2r+2$.*

Consider a set of vertices $Y \subset X$ with $|Y| = 2r-1$. The family H/Y of edges of H contained in Y is an intersecting family. This, together with the stars of the form $H(x)$ with $x \in X-Y$, cover all the edges of H. Hence $\tau_0(H) \le 1 + (n-2r+1) = n-2r+2$.

Theorem 14 (Lovász [1978]). *Let n,r be integers with $2 \le r \le \frac{n}{2}$. We have*

$$\tau_0(K_n^r) = n-2r+2.$$

Proof. From the preceding proposition, it is enough to prove that

$$\tau_0(K_n^r) \geq n-2r+2.$$

Let $d = n-2r$. We argue by contradiction and suppose that we can decompose K_n^r into $d+1 = n-2r+1$ intersecting families $H_1,H_2,...,H_{d+1}$. From a theorem of Gale [1956], for every $k \geq 1$ we can place $d+2k$ points on the sphere $S^d = \{\mathbf{x},\mathbf{x} \in \mathbb{R}^{d+1}, ||\mathbf{x}||=1\}$ in space of $d+1$ dimensions, in such a way that every open hemisphere contains at least k of these points. Hence we can place the $n = d+2r$ vertices of K_n^r on S^d in such a way that every hemisphere contains at least r vertices (and hence at least one edge of K_n^r).

Denote by P_i the set of points $\mathbf{x}$ of the sphere S^d such that the (open) hemisphere centred on $\mathbf{x}$ contains an edge of the family H_i. Since for every point of S^d, the hemisphere centred on this point contains an $E \in K_n^r$ (hence an E belonging to an H_i) we have $S^d = P_1 \cup P_2 \cup \cdots \cup P_{d+1}$.

We now use Borsuk's "antipodal points theorem" [1933] which says that if a sphere $S^d \subset \mathbb{R}^{d+1}$ is the union of $d+1$ open sets, then one of these sets contains two antipodal points. Let set P_{i_0} contain two antipodal points $\mathbf{x}$ and $\mathbf{y}$. The hemisphere of S^d centred on $\mathbf{x}$ contains an edge E belonging to H_{i_0}, and the hemisphere centred on $\mathbf{y}$ contains an edge F belonging to H_{i_0}. Consequently, $E \cap F = \emptyset$. This contradicts the fact that H_{i_0} is an intersecting family.

Exercises on Chapter 4

Exercise 1 (§1)

Determine the chromatic number and the stability number of K_n^r and of $K^r_{n_1,n_2,...,n_r}$.

Exercise 2 (§1)

Let H be a hypergraph on X. Show that if $\alpha(H/A) \geq \frac{|A|}{2}$ for every $A \subset X$, then X can be covered by $\alpha(H)$ edges or singleton vertices. (Lehel [1982]).

Exercise 3 (§1)

Let H be a hypergraph, and let $m_1,m_2,...,m_k$ be positive integers. Show that H is the union of k hypergraphs H_i with no edges in common and with $\chi(H_i) \leq m_i$ if and only if $\chi(H) \leq m_1 m_2 \cdots m_k$. (Miller, Müller [1981]).

Exercise 4 (§1)

Show that if a hypergraph H of rank $r \geq 3$ satisfies $|E \cap E'| \leq r-2$ $(E,E' \in H;\ E \neq E')$, then $\alpha(H) = \alpha$ satisfies $n-\alpha \leq \binom{\alpha}{r-1}$.

Exercise 5 (§1)

On a chess board of $n \times n$ squares we define the "Queen's move hypergraph" H_n^Q as the hypergraph whose vertices are the squares, and for which an edge E_x is the set of squares which a queen placed on square x dominates (including x itself). We define similarly the "King's move hypergraph" H_n^K, etc.

Show that $\chi(H_n^Q) = \chi(H_n^R) = \chi(H_n^B) = \chi(H_n^K) = 2$. ($R$ = rook; B = bishop).

Exercise 6 (§1)

Consider the 3-uniform hypergraph whose vertices are the integers $1,2,...,n$ and whose edges are the triples $\{x,y,z\}$ with $x+y = z$. Show that the stability number of this hypergraph is $[\frac{n}{2}]+1$. (Sedlaček [1970]).

Exercise 7 (§1)

Consider the infinite hypergraph whose vertices are the positive integers, and whose edges are the families of integers forming an arithmetic progression. Show that this hypergraph satisfies the Helly property, and that its chromatic number cannot be 2.

Exercise 8 (§2)

If we associate one of the colours $1,2,...,k$ with each vertex of a hypergraph H, we regard an edge as "strongly coloured" if all its elements have different colours. The *cochromatic number* of H, denoted $\bar{\gamma}(H)$, is the smallest integer k such that for every k-partition $(S_1,S_2,...,S_k)$ (with no empty classes) there exists a strongly coloured edge.

Show that $\bar{\gamma}(K_n^r) = r$.

If H is r-uniform of order n, show that

$$\bar{\gamma}(H) \leq n-r+1.$$

Calculate $\bar{\gamma}(K^r_{n_1,n_2,...,n_r})$.

If G is a graph with p connected components, then $\bar{\gamma}(G) = p+1$. Let G be a graph of order n, and H a hypergraph on the edges of G in which the edges are the cycles of G. Show that $\bar{\gamma}(H) = n$.

Exercise 9 (§2)

Show that between the cochromatic number $\bar{\gamma}(H)$ and the stability number $\alpha(H)$ the following relation holds:

$$\bar{\gamma}(H) \leq \alpha(H) + 1.$$

Show further that for $n \geq p \geq r \geq 2$ there exists an r-uniform hypergraph of order n with $\alpha(H) = p-1$, $\bar{\gamma}(H) = p$. (Sterboul [1975]).

Exercise 10 (§2)

If G is a graph, show that the "product" (cf. Chapter 3, §6) $G \times K_n$ satisfies

$$\bar{\gamma}(G \times K_n) = n(G) + \alpha(G)(n-1) + 1.$$

(Sterboul [1975]).

Exercise 11 (§3)

Show that the vertices of a tree of maximum degree Δ can be uniformly k-coloured for every $k \geq [\frac{\Delta}{2}] + 1$.

Show further that there is a tree with no uniform k-colouring if $k = [\frac{\Delta}{2}]$.

Exercise 12 (§4)

Show that

$$m_k(n,r) \leq \binom{kr-k+1}{r}$$

(Herzog, Schönheim [1972]).

Exercise 13 (§4)

Show that if $p = \frac{n}{k}$ is an integer $\geq r$, then

$$m_k^0(n,r) \geq \binom{n-1}{r-1}\binom{p-1}{r-1}^{-1}.$$

Exercise 14 (§4)

Show that for $n \geq kr$

$$m_k^0(n,r) \leq T(n-1,p-1,r-1)$$

where $p = [n/k]$.

For this, consider an extremal $(r-1)$-uniform hypergraph H_1 of order $n-1$ with no stable set of cardinality $p-1$. Consider the r-uniform hypergraph H_0 of order n obtained by adding to every edge the same additional vertex x_0, and show that H_1 has no uniform k-colouring. (Berge, Sterboul [1977]).

Exercise 15 (§4)

Let H be an r-uniform hypergraph of order n and stability number α. Show that the maximum number of edges containing a set $T \subset X$ with $|T| = r-1$ is an integer z satisfying

$$\alpha + \binom{\alpha}{r-1} z \geq n.$$

Deduce from this that the number m of edges in such a hypergraph satisfies

$$\alpha + \binom{\alpha}{r-1} rm \binom{n}{r-1}^{-1} \geq n$$

(de Caen [1983]).

Hint: Use the inequality (3) that follows Theorem 9.

Exercise 16 (§4)

Let H be an r-uniform hypergraph of order $n = kr$ which has no uniform k-colouring and which has the minimum number of vertices for this condition. Show that H is a star of K_n^r. (Berge, Sterboul [1977]).

Exercise 17 (§5)

Show that there exists an equitable k-colouring of the edges of K_n^r if and only if

$$\left[[\binom{n}{r}\frac{r}{kn}]\frac{n}{r}\right]^* \leq \binom{n}{r}k^{-1} \leq \left[[\frac{r}{kn}\binom{n}{r}]^*\frac{n}{r}\right].$$

Exercise 18 (§7)

Given integers n,k,t with $n > k > t > 0$ and $n+t > 2k$, consider the graph $G(n,k,t)$ on the set of k-tuples taken from a set of n elements, where two k-tuples A and B are joined if and only if $|A \cap B| < t$. Then $\tau_0(K_n^r)$ is the chromatic number of $G(n,r,1)$. Frankl has conjectured that the chromatic number of $G(n,k,t)$ is $T(n,k,t)$ for n sufficiently large, and has proved it for $t = 2$. (Frankl [1985]).

Exercise 19 (§7) Show that

$$\tau_0(H) \leq \max_i m(H/X - E_i) = 1$$

and that equality is possible only if the connected component of the complement of the graph $L(H)$ having maximum degree is either a clique or an odd cycle without chords.

(Use Brooks's Theorem, *Graphs*, Chapter 15).

Exercise 20 (§7)

Show directly that $\tau_0(K_n) = n-2r+2$ for $n \geq 3$.

Chapter 5

Hypergraphs Generalising Bipartite Graphs

1. Hypergraphs without odd cycles

Let H be a hypergraph on X, and let $k \geq 2$ be an integer. A *cycle of length* k is a sequence $(x_1,E_1,x_2,E_2,x_3,...,x_k,E_k,x_1)$ with:

(1) $E_1,E_2,...,E_k$ distinct edges of H;

(2) $x_1,x_2,...,x_k$ distinct vertices of H;

(3) $x_i,x_{i+1} \in E_i$ $(i = 1,2,...,k-1)$;

(4) $x_k,x_1 \in E_k$.

Observe that the sequence (x_1,E_1,x_1) is not considered to be a cycle. A cycle of length k odd (respectively even) is called an odd cycle (respectively even).

Graphs without odd cycles possess such remarkable properties as:

- the Helly property,
- the König property,
- the dual König property,
- the coloured edge property,
- the two-colourability of the vertices.

Is it still true for hypergraphs?

Example: Consider a 0-1 matrix A with p rows and q columns. Let H be the hypergraph whose vertices are the entries of the matrix having value 1, and whose edges are those 1's lying in a single row or a single column. Clearly H is the dual of a bipartite graph G (whose vertices are the rows and columns of the matrix A). Thus H contains no odd cycles, and it is easy to show the existence of a 2-colouring of the vertices of H.

For a stronger statement, call a *B-cycle* a cycle $(x_1,E_1,x_2,E_2,...,E_k,x_1)$ with the following properties:

(1) k is odd;

$$A = \overbrace{\begin{pmatrix} 0 & 0 & 0 & 1^+ & 0 & 1^+ & 0 & 1^- \\ 0 & 1^+ & 0 & 1^- & 0 & 0 & 1^+ & 0 \\ 1^+ & 0 & 0 & 0 & 1^+ & 1^+ & 1^- & 1^+ \\ 1^- & 0 & 0 & 1^+ & 1^+ & 0 & 1^+ & 0 \\ 1^+ & 0 & 0 & 1^+ & 1^- & 1^- & 1^+ & 0 \end{pmatrix}}^{q \text{ columns}} \Bigg\} \; p \text{ rows}$$

Figure 1. *Example of a 2-colouring of H with* $+$, $-$.

(2) $H' = (E_1,E_2,\dots,E_k)$ has maximum degree $\Delta(H') = 2$;

(3) $|E_i \cap E_{i+1}| = 1 \quad (i = 1,2,\dots,k-1)$;

(4) $|E_k \cap E_1| \geq 1$.

Example. The projective plane P_7 and the complete hypergraph K^r_{2r-1}, which are not 2-colourable, contain B-cycles of length 3.

Theorem 1 (Fournier, Las Vergnas [1972], [1984]). *Every non 2-colourable hypergraph contains a B-cycle.*

Proof. Let H be a non 2-colourable hypergraph; by removing the maximum number of edges without altering this property, we may suppose that $\chi(H) > 2$ and $\chi(H-E) = 2$ for each $E \in H$. Suppose that H contains no B-cycle. Let $E_0 \in H$: since $H-E_0$ is 2-colourable, let (A,B) be a 2-colouring of $H-E_0$. Since $\chi(H) > 2$, the edge E_0 is monochromatic, and we may assume $E_0 \subset A$.

Now define one by one the bipartitions (A_1,B_1), (A_2,B_2),... in such a way that the families H_1,H_2,... formed by the monochromatic edges in the different partitions are pairwise disjoint. Since H has only finitely many edges this will imply that for some integer k the family H_k is empty; that is to say, that (A_k,B_k) is a 2-colouring of H: this will contradict $\chi(H) > 2$ and will complete the proof.

Consider a vertex $z \in E_0$, denote by $T_0 = \{z\}$ the singleton z, and set

$$\begin{cases} A_1 = A - T_0 \\ B_1 = B \cup T_0. \end{cases}$$

We have thus defined a new partition (A_1,B_1), and the family H_1 of monochromatic edges in this partition satisfies

(Π_1) H_1 is disjoint from $\{E_0\}$,

(Π_1') every edge of H_1 is contained in B_1 and meets T_0,

(Π_1'') there exists a set $T_1 \in Tr\, H_1$ contained in $B \cap B_1$ and disjoint from E_0 ($Tr\, H$ denotes the transversal hypergraph of H, cf. Ch. 2 §1).

More generally, suppose that we have defined a bipartition (A_{i-1},B_{i-1}) and the associated family H_{i-1} of monochromatic edges. Let $T_{i-1} \in Tr\, H_{i-1}$ be contained in $A \cap A_{i-1}$ (if i is odd) or in $B \cap B_{i-1}$ (if i is even). For $i \geq 1$ odd, set

$$\begin{cases} A_i = A_{i-1} - T_{i-1}, \\ B_i = B_{i-1} \cup T_{i-1}. \end{cases}$$

For $i \geq 2$ even, set

$$\begin{cases} A_i = A_{i-1} \cup T_{i-1}, \\ B_i = B_{i-1} - T_i - 1. \end{cases}$$

We shall now show by induction on i that the family H_i of monochromatic edges with respect to the bipartition (A_i,B_i) satisfies the following three properties:

(Π_i) H_i is disjoint from the families $H_0 = \{E_0\}$, $H_1,H_2,\dots,H_{i-1}$;

(Π_i') every edge of H_i is contained in A_i (for i even) or in B_i (for i odd), and meets T_{i-1};

(Π_i'') there exists a set $T_i \in Tr\, H_i$ contained in $A \cap A_i$ (for i even) or $B \cap B_i$ (for i odd), and which meets none of the sets $E_0 - T_0$, T_0, T_1, $T_2,\dots,T_{i-1}$.

Let $k > 1$ be an integer; assume first that k is odd. Suppose that we have shown Π_i, Π_i', Π_i'' for each $i \leq k-1$.

1) Proof that Π_k holds.

From $\Pi_1'',\Pi_2'',\ldots,\Pi_{k-1}''$ the sets $E_0-T_0,T_0,\ldots,T_{k-1}$ are pairwise disjoint (and each set has changed colour completely in a single step in the procedure). Hence $T_i \subset B_k$ for i even $\leq k-1$, or $T_i \subset A_k$ for i odd $\leq k-1$. For $i \leq k-1$ every edge of H_i meets T_i (since $T_i \in Tr\, H_i$) and T_{i-1} (from Π_i') and cannot be monochromatic with (A_k,B_k) since $T_0 \subset B_k$ and $E-T_0 \subset A_k$. Thus the family H_k of monochromatic edges has no edge in common with $\{E_0\}$, $H_1,H_2,\ldots,H_{k-1}$.

2) Proof that Π_k' holds.

Every edge of H_k is 2-coloured in (A_{k-1},B_{k-1}) from Π_k, and is monochromatic with (A_k,B_k); thus it must meet T_{k-1}, which is the set of vertices which change colour in the k-th step, and is contained in B_k.

3) Proof that Π_k'' holds.

Since k is odd, the edges of H_k are contained in B_k (from Π_k'): thus there exists a $T_k \in Tr\, H_k$ contained in B_k. Further no edge of H_k is contained in $T_0 \cup T_2 \cup \cdots \cup T_{k-1}$ since such an edge would be monochromatic with (A,B), which contradicts Π_k. Thus we may assume T_k is contained in $B_k - (T_0 \cup T_2 \cup \cdots \cup T_{k-1}) = B_k \cap B$. From $\Pi_1'',\Pi_2'',\ldots,\Pi_{k-1}''$ the sets $T_0,T_2,T_4,\ldots,T_{k-1}$ are contained in A; thus they do not meet T_k. By the definition of the transformation, the sets $E_0-T_0,T_1,T_3,\ldots,T_{k-2}$ are contained in A_k, and so they do not meet T_k.

4) If we now suppose k is even, nothing changes in the above argument except for one point: the edges of H_k are contained in A_k and $T_k \subset A \cap A_k$. Consequently T_k does not meet $T_0,T_2,\ldots,T_{k-2}$ (which are contained in B_k) or $T_1,T_3,\ldots,T_{k-1}$ (which are contained in B), but it remains to show that T_k does not meet E_0-T_0.

More precisely, we shall show that every edge of H_k is disjoint from E_0-T_0. Otherwise, there exists an $E_k \in H_k$ which meets E_0-T_0: let $x_0 \in E_k \cap (E_0-T_0)$. From Π_k' there exists a vertex $x_k \in T_{k-1} \cap E_k$, and by the minimality of the transversal T_{k-1}, there exists an edge $E_{k-1} \in H_{k-1}$ such that $E_{k-1} \cap T_{k-1} = \{x_k\}$; since $E_k \subset A_k$ and $E_{k-1} \subset B_k \cup T_{k-1}$ we have also $E_{k-1} \cap E_k = \{x_k\}$. Repeating this procedure with E_{k-1}, etc., we obtain a sequence term by term

$$E_0 - T_0,\ x_0,\ E_k,\ x_k,\ E_{k-1},\ x_{k-1},\dots,E_1,\ x_1 = z,$$

with, for $i = 1,2,\dots,k$, the relations

$$(1) \qquad E_i \in H_i,\ x_i \in T_{i-1},\ E_i \cap E_{i-1} = \{x_i\}.$$

Then the sequence $(x_0,E_0,x_1,E_1,x_2,\dots,x_k,E_k,x_0)$ defines a cycle and satisfies $|E_i \cap E_{i-1}| = 1$ for each $i \geq 1$; further its length $k+1$ is odd.

So, by virtue of the hypothesis, there exists a vertex y of degree > 2 in the hypergraph $H' = (E_0,E_1,\dots,E_k)$; suppose for example:

$y \in E_p \cap E_q \cap E_r$

$0 \leq p < q < r \leq k$

$r-p$ minimum.

We shall show first that $y \neq x_{p+1},x_{p+2},\dots,x_r$. Indeed, if for example r is even, then $E_r \subset A_r$ from Π'_r, so the vertex y is different from $x_1,x_3,\dots,x_{r-1}$ (which are in B_r, from (1) and $\Pi''_1,\Pi''_2,\ \dots\ ,\Pi''_{r-1}$). If $y = x_s$ for s even, $p+1 \leq s < r$, then the cycle $(x_s,E_s,x_{s+1},\dots,x_r,E_r,x_s)$ is an odd cycle of maximum degree 2 (by the minimality of $r-p$), so it is a B-cycle, contradicting the hypothesis. If $y = x_r$, then $r = q+1$ from (1) and the minimality of $r-p$. Hence q is odd. Moreover, p is odd (since if p were even, $T_p \subset A_p$, and does not contain x_r which is in B_p). Hence the cycle $(x_r,E_p,x_{p+1},E_{p+1},\dots,x_q,E_q,x_r)$ is odd of maximum degree 2; so it is a B-cycle, which contradicts the hypothesis.

Observe that the indices p,q have different parities: otherwise the cycle $(y,E_p,x_{p+1},E_{p+1},\dots,x_q,E_q,y)$ is odd of maximum degree 2, so it is a B-cycle, contradicting the hypothesis. Similarly, the indices q and r have different parities. Suppose for example p even, q odd, r even. Then $E_p \subset A_p$, $E_q \subset B_q$, $E_r \subset A_r$. Since $E_q \cap B \subset B_q$ we have $E_p \cap E_q \subset A$. For the same reason, $E_q \cap E_r \subset B$, which implies that $E_p \cap E_q \cap E_r = \emptyset$ and the contradiction follows.

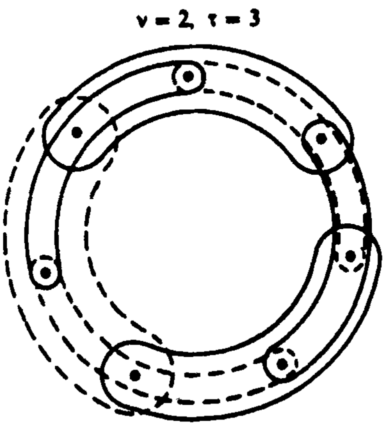
Corollary 1. *In a non 2-colourable hypergraph of rank ≤ 3, there exists a B-cycle such that every pair of two non-consecutive edges are disjoint.*

Let H be a hypergraph with $\chi(H) \geq 3$, $r(H) \leq 3$. We may suppose that we have removed from H as many edges as possible without it becoming 2-colourable. Now, from Theorem 1, there exists a B-cycle $(x_1,E_1,x_2,\dots,E_k,x_1)$ which we may suppose of minimum length k. If $E_1 \cap E_j \neq \emptyset$ for an integer j, $3 \leq j \leq k-1$, then there exists a vertex $y \in E_1 \cap E_j$. Since the degree of the B-cycle is 2, the vertex y is distinct from $x_1,x_2,\ldots,x_k$; and since H has rank ≤ 3, we have $E_1 = \{y,x_1,x_2\}$ and $E_j = \{y,x_j,x_{j+1}\}$. One of the two cycles $(y,E_1,x_2,\ldots,E_j,y)$ and $(x_1,E_1,y,E_j,x_{j+1},\ldots,E_k,x_1)$ is odd. Since $E_1 \cap E_j = \{y\}$, this cycle is a B-cycle, which contradicts the minimality of k.

Corollary 2. *In a non 2-colourable hypergraph, there is an odd cycle of maximum degree 2 such that every pair of two non-consecutive edges are disjoint.*

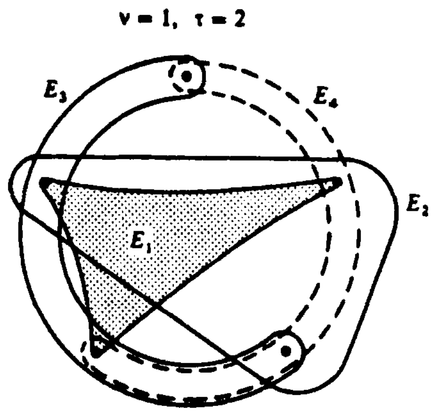
(Same proof, replacing each occurrence of "B-cycle" by "odd cycle of maximum degree 2").

Following these results, we might expect that hypergraphs without odd cycles of maximum degree 2 would have those properties apparent in bipartite graphs; however, they may satisfy the Helly property as in Figure 2, or not, as in Figure 3.



Hypergraph without odd cycles of maximum degree 2 (with the Helly property)

Figure 2.



Hypergraph without odd cycles of maximum degree 2 (without the Helly property)

Figure 3.

From these results, we obtain the following characterization of the hypergraphs which contain no odd cycle:

Theorem 2. *A hypergraph* $H = (E_1,E_2,...,E_m)$ *has no odd cycles if and only if every hypergraph* $H' = (E'_1,E'_2,...,E'_m)$ *with* $E'_i \subset E_i$ *for each* i *is 2-colourable.*

Proof. If H contains no odd cycles, Theorem 1 shows that $\chi(H) \leq 2$. The hypergraph H' is also without odd cycles, so $\chi(H') \leq 2$.

If H contains an odd cycle $(x_1,E_1,x_2,E_2,...,E_k,x_1)$, then there exists a hypergraph H' of the form indicated which has edges $[x_1,x_2],[x_1,x_3],...,[x_k,x_1]$, whence $\chi(H') > 3$. Contradiction.

The class of hypergraphs without odd cycles has been studied from the point of view of matrices by Commoner [1973]; Yannakakis [1985] has given a polynomial algorithm to test whether a given hypergraph is in this class.

Theorem 3. *A hypergraph* $H = (E_1,E_2,...,E_m)$ *is cycle-free if and only if for every non-empty subset* J *of* $\{1,2,...,m\}$, *we have*

$$(1) \qquad |\bigcup_{j\in J} E_j| > \sum_{j\in J} (|E_j|-1)\,.$$

Proof.

1. If H contains a cycle $(a_1,E_1,a_2,E_2,...,E_k,a_1)$ we obtain, setting $K = \{1,2,...,k\}$

$$|\bigcup_{j\in K} E_j| = |\bigcup_{j\in K}(E_j-\{a_j\})| \leq \sum_{j\in K} |E_j-\{a_j\}| = \sum_{j\in K} (|E_j|-1)$$

Thus condition (1) fails.

2. If H contains no cycles, the partial hypergraph $H' = (E_j/j\in J)$ also contains no cycles. Set $\bigcup_{j\in J} E_j = \{x_i/i\in I\}$ and form the bipartite graph G on $I \cup J$, where $i \in I$ and $j \in J$ are adjacent if and only if $x_i \in E_j$.

Since G contains no cycles, we have $m(G) < n(G)$ (cf. *Graphs*, Ch. 2); thus

$$\sum_{j \in J} |E_j| = m(G) < n(G) = |\bigcup_{j \in J} E_j| + |J|$$

whence (1) holds.

Remark. If H is k-uniform, a necessary and sufficient condition for H to have no cycles is that for every non-empty subset J of $\{1,2,\ldots,m\}$,

$$|\bigcup_{j \in J} E_j| > (k-1)|J|.$$

We may generalise this result in the following way:

Generalisation (Las Vergnas, [1970]). *Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph and let $k \geq 2$ be an integer; a necessary and sufficient condition for the existence of a k-uniform hypergraph H' without cycles, $H' = (E'_1, E'_2, \ldots, E'_m)$ with $E'_i \subset E_i$ for every i, is that*

$$|\bigcup_{j \in J} E_j| > (k-1)|J| \qquad (J \neq \emptyset)$$

Between the class of hypergraphs without cycles and the class of hypergraphs without B-cycles, there are many classes, each having interesting characteristics and concrete combinatorial applications. In this chapter we shall study the classes of hypergraphs shown in Figure 4.

2. Unimodular Hypergraphs

A matrix $A = ((a_j^i))$ is said to be *totally unimodular* if every square submatrix of A has determinant equal to 0, $+1$ or -1. A hypergraph is said to be *unimodular* if its incidence matrix is totally unimodular.

It is immediate from this definition that *the dual, the subhypergraphs and the partial hypergraphs of a unimodular hypergraph are unimodular.*

A combinatorial property of unimodular hypergraphs is revealed in the concept of an "equitable colouring".

Theorem 4. *A hypergraph H on X is unimodular if and only if for every $S \subset X$ the subhypergraph H_S has an equitable 2-colouring: that is to say a bipartition (S_1, S_2) of S such that each edge E of H_S satisfies*

$$\left[\frac{|E|}{2}\right] \leq |E \cap S_i| \leq \left[\frac{|E|}{2}\right]^* \qquad (i = 1,2).$$

Proof. If an $n \times m$ matrix $A = ((a_j^i))$ is totally unimodular, it is clear that $a_j^i = 0,+1$ or -1 (since the value of each entry is a determinant of order 1 from A); further Ghouila-Houri [1962] showed that A is totally unimodular if and only if every non-empty set $I \subset \{1,2,\ldots,n\}$ may be partitioned into two disjoint sets I_1 and I_2 in such a

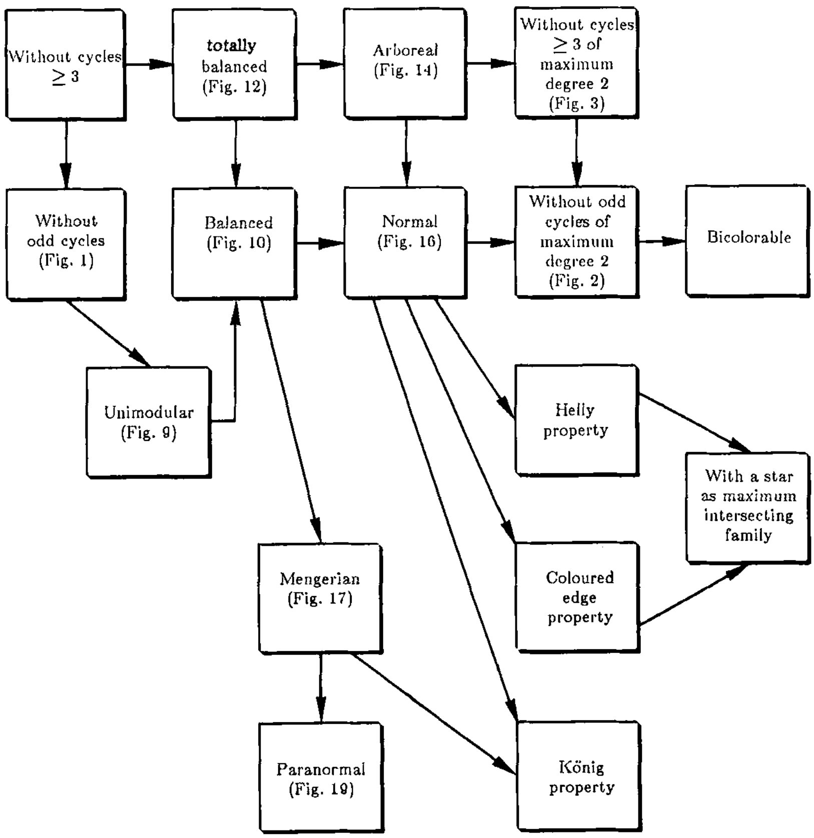


Figure 4. ***Implication scheme for the principal classes of hypergraphs generalizing trees and bipartite graphs.***

way that

$$|\sum_{i\in I_1} a^i_j - \sum_{i\in I_2} a^i_j | \leq 1 \qquad (j \leq m).$$

If A is the incidence matrix of a hypergraph we obtain the required 2-colouring with $S_1 = \{x_i / i \in I_1\}$, $S_2 = \{x_i / i \in I_2\}$.

Example 1. Bipartite multigraph.

Let G be a bipartite multigraph; clearly every subgraph of G is a bipartite multigraph, and hence is 2-colourable. Thus G is a unimodular hypergraph.

Example 2. Interval hypergraph.

Let H be defined by a set of points on a line and a family of intervals. Clearly for $A \subset X$ the subhypergraph H_A is an interval hypergraph, for which we obtain an equitable 2-colouring by successively colouring the points from left to right red and blue alternately. Thus H is unimodular.

Example 3. Hypergraph of paths in an oriented tree.

Let T be a tree on a set X with a (unique) orientation on each edge. Let H be a hypergraph on X such that each edge is an oriented path of T. Clearly a 2-colouring of T defines an equitable 2-colouring of H (cf. Figure 5). Every subhypergraph of H also has an equitable 2-colouring: if we remove a vertex a from H, consider the tree T' of Figure 6 for which every 2-colouring induces an equitable 2-colouring of $H_{X-\{a\}}$.

Example 4. Hypergraph on the arcs of a tree.

Let T_0 be a tree on a set X, with a unique orientation on each edge, which defines a set U of arcs. Let H_0 be a hypergraph on U such that each edge is a set of arcs forming a path of T_0. Clearly we may colour the arcs of T_0 in 2 colours, $+$ and $-$, in such a way that every pair of consecutive arcs contain both colours (cf. Figure 7); this defines an equitable 2-colouring of H_0. Every subhypergraph also has an equitable 2-colouring: if we remove an arc u of U, consider the tree T_0' of Figure 8 for which a 2-colouring induces an equitable 2-colouring of $H_{U-\{u\}}$.

Theorem 5. *Every hypergraph without odd cycles is unimodular.*

Proof. Since no subhypergraph of a hypergraph without odd cycles contains an odd cycle, it suffices to show that a hypergraph H without odd cycles may be equitably 2-coloured.

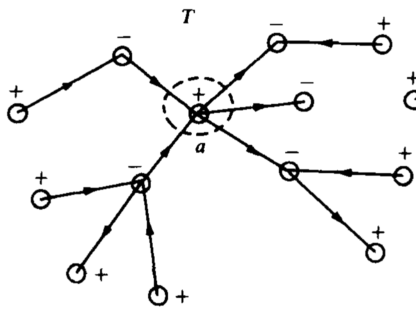


Figure 5

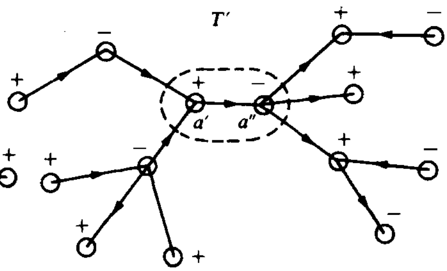


Figure 6

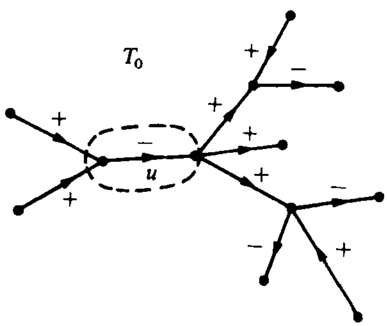


Figure 7

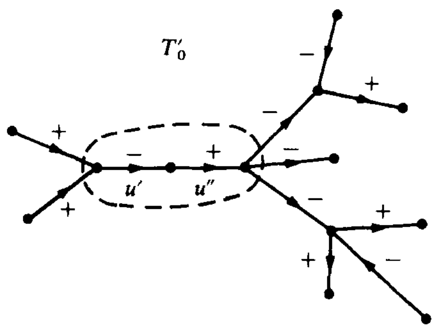


Figure 8

For $i \leq m$, put $r_i = |E_i|$ and define a map $y_i:\{1,2,\ldots,r_i\} \rightarrow X$ so that

$$E_i = \{y_i(1),y_i(2),\ldots,y_i(r_i)\}$$

Consider the set $\mathcal{F}_i$ of the following pairs:

$$y_i(1)y_i(2),$$

$$y_i(3)y_i(4)$$

$$\vdots$$

$$y_i(2[r_i/2]-1)y_i(2[r_i/2]).$$

The union of the $\mathcal{F}_i$'s is a graph G and we may suppose that the y_i's (and hence the $\mathcal{F}_i$'s) have been chosen so that the minimum length of an odd cycle of G is as small as possible. If G has odd cycles, consider an odd cycle of G of minimum length, say $\mu = [a_1, a_2, \ldots, a_1]$. The cycle μ is elementary. We shall show that μ does not contain two edges from the same set $\mathcal{F}_i$.

Indeed, if for example $[a_s, a_{s+1}] \in \mathcal{F}_i$ and $[a_t, a_{t+1}] \in \mathcal{F}_i$, by replacing these two edges in $\mathcal{F}_i$ by the edges $[a_s, a_{t+1}]$ and $[a_t, a_{s+1}]$, the graph G' so obtained has an odd cycle which is shorter than μ, (as one of the two sequences $[a_1, a_2, \ldots, a_s, a_{t+1}, \ldots, a_1]$ and $[a_{s+1}, a_{s+2}, \ldots, a_t, a_{s+1}]$ is odd) which contradicts the definition of G. Further, if the cycle μ has its edges in different classes $\mathcal{F}_i$ then it defines an odd cycle of H, which contradicts our hypothesis that H has no odd cycles. Thus such a cycle μ cannot exist.

Since the graph G has no odd cycles, there exists a 2-colouring (S_1, S_2) of its vertices: this constitutes also an equitable 2-colouring for H.

Theorem 6 (de Werra [1971]). *A unimodular hypergraph H has an equitable k-colouring for every $k \geq 2$.*

Proof. For $k = 2$ the statement follows from Theorem 4. For $k > 2$ consider a partition $(S_1, S_2, \ldots, S_k)$ of the vertices of H into k classes. For $i, j \leq k$ and for $E \in H$ put

$$\epsilon_{ij}(E) = |S_i \cap E| - |S_j \cap E|$$

$$\epsilon(E) = \max_{i,j} \epsilon_{ij}(E).$$

Clearly $\epsilon(E) \geq 0$. If $\epsilon(E) \leq 1$ for every $E \in H$, the partition is an equitable k-colouring of the hypergraph H, and vice versa. Suppose therefore that there is an edge E_0 with $\epsilon(E_0) \geq 2$ and let p, q be indices for which $\epsilon_{pq}(E_0) = \epsilon(E_0)$. Then

$$|S_q \cap E_0| \leq |S_i \cap E_0| \leq |S_p \cap E_0| \quad (i \neq p,q)$$

The subhypergraph of H induced by the set $S_p \cup S_q$ admits an equitable 2-colouring (S'_p, S'_q). Put $S'_i = S_i$ for $i \neq p,q$. The new partition $(S'_1, S'_2, \ldots, S'_k)$ defines new coefficients ϵ'_{ij}, such that every $E \in H$ satisfies $\epsilon'_{pq}(E) \leq 1$. Furthermore

$$\epsilon'_{ij}(E) = \epsilon_{ij}$$

for i and $j \neq p,q$. Further, for $i \neq p,q$ we cannot have $\epsilon'_{ip}(E) = \epsilon(E_0)$ unless

$\epsilon_{ip}(E) = \epsilon(E_0)$ or $\epsilon_{iq}(E) = \epsilon(E_0)$.

In summary, the number of triples (r,s,E) with $\epsilon_{rs}(E) = \epsilon(E_0)$ has decreased by at least one. By repeating this transformation we finally obtain a partition with $\epsilon'(E) \leq 1$ for each $E \in H$; this partition is an equitable k-colouring of H.

Corollary 1. *Let H be a unimodular hypergraph and let $k = \min_{E \in H} |E|$; there exists a partition $(T_1,T_2,...,T_k)$ of the set X of vertices of H into k transversal sets such that, for every $E \in H$,*

$$(1) \qquad \left[\frac{1}{k}|E|\right] \leq |E \cap T_i| \leq \left[\frac{1}{k}|E|\right]^* \quad (i = 1,2,...,k)$$

Indeed, H admits and equitable k-colouring $(T_1,T_2,...,T_k)$, and consequently (1) holds. Further, as $k = min\ |E|$, each T_i is a transversal.

Corollary 2. *Let H be a unimodular hypergraph and let $k \geq 1$; then there exists a decomposition $H = H_1+H_2+ \cdots +H_k$ into k classes such that for every vertex x of H,*

$$\left[\frac{1}{k}\ d_H(x)\right] \leq d_{H_i}(x) \leq \left[\frac{1}{k}\ d_H(x)\right]^* \quad (i = 1,2,...,k).$$

Indeed, apply Theorem 6 to the dual hypergraph H^*, which is also unimodular.

Corollary 3. *Every unimodular hypergraph satisfies the coloured edge property.*
Indeed, set $k = \Delta(H)$ in Corollary 2.

Our interest in totally unimodular matrices arises principally from the following result:

Theorem 7 (Hoffman, Kruskal [1956]). *Let A be an $n \times m$ matrix: the following conditions are equivalent:*

(i) *A is totally unimodular;*

(ii) *for every $\mathbf{c} \in \mathbb{Z}^n$ the polyhedron of c-matchings*

$$Q(\mathbf{c}) = \{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ \mathbf{y} \geq \mathbf{0},\ A\mathbf{y} \leq \mathbf{c}\}$$

has only integer valued extreme points,

(iii) *for every* $\mathbf{b},\mathbf{c} \in \mathbb{Z}^n$, *for every* $\mathbf{p},\mathbf{q} \in \mathbb{Z}^m$, *the set*

$$Q(\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q}) = \{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ \mathbf{b} \leq A\mathbf{y} \leq \mathbf{c};\ \mathbf{p} \leq \mathbf{y} \leq \mathbf{q}\}$$

is empty or contains an integer valued point.

Proof.

(i) implies (ii). Indeed, the extreme points of the polyhedron $Q(\mathbf{c})$ are given by the intersections of planes of the form $<\mathbf{a}^i,\mathbf{y}> = c_i$. Cramer's rule says that every solution $\mathbf{y}$ of such a system has for each coordinate y_i the quotient of two determinants; the first is integer valued (as a^i_j is an integer), the second has value 0 or ± 1 (since A is unimodular). Thus the point $\mathbf{y}$ has all its coordinates integer valued.

(ii) implies (i). Let B be a regular square submatrix of order n of the matrix

$$(A,I_n) = \left(\begin{array}{c|ccccc} & 1 & & & & 0 \\ & & 1 & & & \\ A & & & 1 & & \\ & & & & 1 & \\ & 0 & & & & 1 \end{array}\right)$$

Let $\mathbf{y} \in \mathbb{Z}^n$ be such that $\mathbf{y} + B^{-1}\mathbf{u}^i \geq 0$, where $\mathbf{u}^i$ is the ith unit vector of $\mathbb{Z}^n$. The vector $\mathbf{z} = \mathbf{y} + B^{-1}\mathbf{u}^i$ satisfies $B\mathbf{z} = B\mathbf{y} + \mathbf{u}^i \in \mathbb{Z}^n$. Consequently $\mathbf{z}$ defines the non-zero components of an extreme point of $Q(\mathbf{c})$ where $\mathbf{c} = B\mathbf{y} + \mathbf{u}^i$; thus, from (ii), $\mathbf{z} \in \mathbb{Z}^n$.

Therefore $B^{-1}\mathbf{u}^i = \mathbf{z} - \mathbf{y} \in \mathbb{Z}^n$ for $i = 1,2,\ldots,n$ and thus the matrix B^{-1} has integer coefficients. Hence $det\, B$ and $det\, B^{-1}$ are integers which satisfy $(det\, B)(det\, B^{-1}) = det\, I_n = 1$. Thus $det\, B = \pm 1$. This proves that A is totally unimodular. (The idea for this much simpler proof is due to Viennot and Dantzig, [1968]).

(ii) is equivalent to (iii). The total unimodularity of A is equivalent to the total unimodularity of the matrix

$$\overline{A} = \begin{pmatrix} A \\ -A \\ I_m \\ -I_m \end{pmatrix}$$

We now apply (ii) with $\mathbf{c} = (c_1, c_2, \ldots, c_n, -b_1, -b_2, \ldots, -b_n, q_1, q_2, \ldots, q_m, -p_1, -p_2, \ldots, -p_m)$ and the matrix $\overline{A}$. Thus, (iii) follows.

To show that this result implies all the characterisations of unimodular matrices by forbidden structures such as those of Ghouila-Houri [1962] quoted above, or those of Camion [1965], etc., the reader is referred to the excellent exposé by Padberg [1988].

Consider a hypergraph H, and its incidence matrix $A = ((a_j^i))$ (with n rows, m columns, with no zero rows or zero columns). Let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n$. A *c—matching* is a vector $\mathbf{y}$ with integer coordinates of the polyhedron

$$Q(\mathbf{c}) = \{\mathbf{y} / \mathbf{y} \in \mathbb{R}^m,\ \mathbf{y} \geq 0,\ A\mathbf{y} \leq \mathbf{c}\}$$

For $\mathbf{c} = \mathbf{1}$, a point of $Q(\mathbf{c})$ with integer coordinates is necessarily 0-1 valued, and a 1-matching is nothing but a matching. If we associate with each edge E_j an integer $d_j \geq 0$ called the *weight* of the edge E_j, and if $\sum_{j=1}^{m} d_j y_j$ is the total weight of the c-matching $\mathbf{y}$, we may ask for the *maximum weight* of a c-matching, which we denote by

$$N - \max_{y \in Q(c)} <\mathbf{d},\mathbf{y}> = \max\{<\mathbf{d},\mathbf{y}> / \mathbf{y} \in Q(\mathbf{c}) \cap \mathbb{N}^m\}$$

In particular, if $\mathbf{c} = \mathbf{1}$, $\mathbf{d} = \mathbf{1}$ we have $N - \max <\mathbf{d},\mathbf{y}> = \nu(H)$.

For a vector $\mathbf{d} \in \mathbb{N}^n$ we may define a **d**-*transversal* to be a vector $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ with integer coordinates of the polyhedron

$$P(\mathbf{d}) = \{\mathbf{t} / \mathbf{t} \in \mathbb{R}^n,\ \mathbf{t} \geq \mathbf{0},\ A^* \mathbf{t} \geq \mathbf{d}\}.$$

Defining the *cost* of a vertex x_i of the hypergraph to be an integer $c_i \geq 0$, we may ask for the minimum cost $\sum_{i=1}^{n} c_i t_i$ of a transversal $\mathbf{t}$, which we denote by

$$N\text{-}\min_{t \in P(d)} \langle \mathbf{c},\mathbf{t} \rangle = \min\{\langle \mathbf{c},\mathbf{t}\rangle / \mathbf{t} \in P(\mathbf{d}) \cap \mathbb{N}^n\}$$

In particular, $N\text{-}\min_{A^*t \geq 1} \langle \mathbf{1},\mathbf{t} \rangle = \tau(H)$.

We may now state, as an application of Theorem 7:

Corollary 1. *Let H be a unimodular hypergraph with n vertices and m edges; for $\mathbf{c} \in \mathbb{N}^n$ and $\mathbf{d} \in \mathbb{N}^m$, we have*

$$N\text{-}\max_{y \in Q(c)} \langle \mathbf{d},\mathbf{y} \rangle = N\text{-}\min_{t \in P(d)} \langle \mathbf{c},\mathbf{t} \rangle$$

Proof. If H is unimodular, the maximum of $\langle \mathbf{d},\mathbf{y} \rangle$ for $\mathbf{y} \in Q(\mathbf{c})$ is attained at a point $\mathbf{y}_0$ having integer coordinates; the minimum of $\langle \mathbf{c},\mathbf{t} \rangle$ for $\mathbf{t} \in P(\mathbf{d})$ is attained at a point $\mathbf{t}_0$ having integer coordinates. The duality theorem of linear programming shows that

$$\langle \mathbf{d},\mathbf{y}_0 \rangle = \langle \mathbf{c},\mathbf{t}_0 \rangle.$$

This implies the equality stated in Corollary 1.

Corollary 2. *A unimodular hypergraph H of rank r can be strongly coloured with r colours.*

Proof. Let $A = ((a_j^i))$ be the incidence matrix of H, where the rows represent the vertices and the columns represent the edges. An n-dimensional vector $\mathbf{z}$ with coordinates 0 or 1 is the characteristic vector of a set $S \subset X$, and $|S \cap E_j|$ is equal to the scalar product $\langle \mathbf{z},\mathbf{a}_j \rangle$.

There exists a set S which meets each edge E_j at most once, and exactly once each edge with $|E_j| = r$, if and only if there exists an integer solution to the following system of inequalities:

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{1}$$

$$0 \leq \langle \mathbf{z},\mathbf{a}_j \rangle \leq 1 \quad \text{if } E_j \in H$$

$$1 \leq \langle \mathbf{z},\mathbf{a}_j \rangle \leq 1 \quad \text{if } E_j \in H, \text{ and } |E_j| = r.$$

The vector $\mathbf{z} = (\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})$ satisfies all these inequalities, so there exists a solution in integers (and hence in 0,1), which is the characteristic vector of a set S of vertices defining the first colour. Repeating the procedure with the unimodular hypergraph

H_{X-S}, or rank $r-1$, define the second colour S', etc. When we arrive at a hypergraph of rank 1 we have defined a strong colouring $(S, S', \ldots)$ of H with r colours.

Remark. A polynomial time algorithm to test whether a matrix is totally unimodular results from the work of Seymour [1980], and from the extensions of Bixby, Truemper, Tamir, etc. Indeed, the problem of testing if a matrix A is totally unimodular is equivalent to that of testing if its associated matroid $M(A)$ is regular. (For an exposition of the algorithm, cf. Bixby [1982]). For good algorithms to find maximum matchings in certain classes of unimodular hypergraphs, cf. Conforti, Cornuéjols [1987].

3. Balanced Hypergraphs

A hypergraph is said to be *balanced* if every odd cycle has an edge containing three vertices of the cycle. A hypergraph is said to be *totally balanced* if every cycle of length ≥ 3 has an edge containing three vertices of the cycle.

In other words, H is balanced if and only if its incidence matrix contains no square submatrix of the form:

$$B_k = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & . & 0 \\ 0 & 1 & 1 & 0 & \cdots & . & 0 \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & 0 \\ . & & & & & 1 & 0 \\ 0 & . & . & . & 0 & 1 & 1 \end{pmatrix}$$

where $k \geq 3$ is odd. Similarly H is totally balanced if and only if A contains no submatrix B_k with $k \geq 3$.

A totally balanced hypergraph is thus balanced; it is easy to see (by considering all the cycles) that the hypergraphs in Figures 9 and 10 are balanced.

Proposition 1. *Every partial subhypergraph of a totally balanced hypergraph (resp. balanced) is totally balanced (resp. balanced).*

Indeed, if H has A as its incidence matrix, a partial subhypergraph has a submatrix A' of A as its incidence matrix; then if $A' \supset B_k$ we must have $A \supset B_k$.

Proposition 2. *The dual of a totally balanced (resp. balanced) hypergraph is totally*

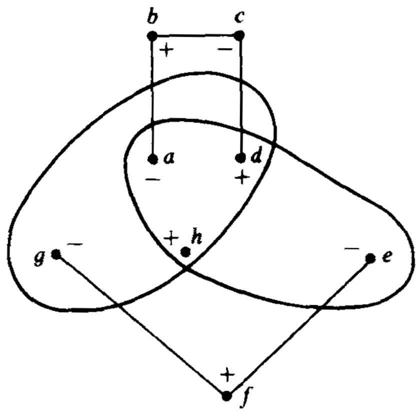


Figure 9. *Balanced hypergraph (strongly unimodular).*

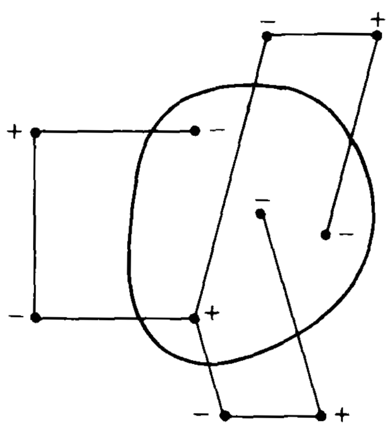


Figure 10. *Balanced hypergraph (not unimodular).*

balanced (resp. balanced).

Indeed, if H has A as its incidence matrix, the dual H^* has for its incidence matrix the transpose A^* of A. Then if $A^* \supset B_k$ we must have $A \supset (B_k)^* = B_k$.

Example 1. Unimodular hypergraphs.

We shall show that every unimodular hypergraph is balanced. Let H be a unimodular hypergraph which is not balanced: the incidence matrix A contains a submatrix of the form B_k with $k \geq 3$ odd. However the matrix B_k is not totally unimodular (since the hypergraph which it represents is an odd cycle C_k, which cannot be equitably 2-coloured and thus cannot be unimodular from Theorem 3). Thus H cannot be unimodular: a contradiction.

Observe that the converse is not true: the hypergraph of Figure 10 is clearly balanced, but it cannot be 2-coloured equitably (because of edge E_1) and thus is not unimodular.

It was precisely in order to generalise some theorems for totally unimodular matrices that the concept of a balanced hypergraph was introduced (Berge [1969], [1972]).

Example 2. Strongly unimodular hypergraphs (Crama, Hammer, Ibaraki [1985]).

Another balanced hypergraph, due to Crama, Hammer and Ibaraki [1985] is the *strongly unimodular* hypergraph: this is a balanced hypergraph which further admits *no odd cycles having one edge containing exactly three vertices of the cycle and all the other containing exactly two vertices of the cycle.* (For example, the hypergraph of Figure 9, which contains odd cycles of length 5 and 7, is strongly unimodular). In other words, H is strongly unimodular if and only if its incidence matrix contains no square submatrix of the form B_k with $k \geq 3$ odd, nor of the form B'_k, where B'_k is obtained from B_k by replacing a 0 by a 1. Consequently we see as before that if H is strongly unimodular then its dual and its partial subhypergraphs are strongly unimodular.

The same authors have shown further that in a strongly unimodular hypergraph H there exists a non-empty set $S \subset X$ meeting each edge of H which is not a loop in 0 or 2 vertices. In Figure 9 we find for example the set $S = \{a,b,c,d\}$. We may show that H is unimodular as follows: consider such a set S_1; then in H_{X-S_1} (which is also strongly unimodular) consider such a set S_2; then in $H_{X-S_1-S_2}$ such a set S_3 etc. Each subhypergraph H_{S_i} being a bipartite multigraph we can colour it equitably with two colours, red and blue. When all the vertices of H have been coloured, the blue set and the red set constitute an equitable 2-colouring of H. Thus from Theorem 3, H is unimodular.

Example 3. Hypergraph of neighbourhoods.

Let T_0 be a tree on $X = \{x_1,x_2,...,x_n\}$; denote by $\mu[x_i,x_j]$ the (unique) path whose extremities are x_i and x_j, and denote by $d(x_i,x_j)$ the distance between x_i and x_j, that is to say the length of $\mu[x_i,x_j]$. For $\rho \geq 0$ we define *the neighbourhood centred at* $a \in X$ of radius ρ to be the set

$$T = \{x / x \in X,\ d(x,a) \leq \rho\}$$

A family $H = (T_1,T_2,...,T_m)$ of neighbourhoods is a hypergraph; we shall show that it is totally balanced.

Indeed, otherwise there exists an odd cycle, say

$$\sigma = (x_1,T_1,x_2,T_2,...,x_k,T_k,x_{k+1} = x_1),$$

such that the set

$$T_i = \{x / x \in X,\ d(x,a_i) \leq \rho_i\}$$

does not contain x_j for $j \neq i,i+1$.

Since $T_i \cap T_{i+1} \neq \emptyset$, we have

$$d(a_i,a_{i+1}) \leq \rho_i + \rho_{i+1},$$

$$d(a_i,x_i) \leq \rho_i,$$

$$d(a_i,x_{i+1}) \leq \rho_i.$$

It is easy to see that in the tree T_0, at least three of the paths $\mu[a_i,a_{i+1}]$ have a non-empty intersection. Let $y \in T_0$ be such that it appears in, say, $\mu[a_1,a_2]$, $\mu[a_p,a_{p+1}]$, $\mu[a_q,a_{q+1}]$. Suppose further that $d(y,x_1) \geq d(y,x_p) \geq d(y,x_q)$.

As $y \in \mu[a_1,a_2]$, we have either $y \in \mu[a_1,x_1]$ or $y \in \mu[x_1,a_2]$.

Suppose, for example, that $y \in \mu[a_1,x_1]$. Then

$$\begin{aligned} 0 \leq \rho_1 - d(a_1,x_1) &= \rho_1 - d(a_1,y) - d(y,x_1) \\ &\leq \rho_1 - d(a_1,y) - d(y,x_p) \\ &\leq \rho_1 - d(a_1,x_p). \end{aligned}$$

Hence $x_p \in T_1$. For the same reason, $x_q \in T_1$. Thus T_1 contains at least three vertices of the cycle σ: contradiction.

Example 4 (Tamir [1985]).

Consider a tree T on $X = \{x_1,x_2,...,x_n\}$ and let $S \subset X$ with $|S| = k$. We may generalise the example 3 by considering for every vertex $x \in X$ the sequence $0 = d_0^x \leq d_1^x \leq d_2^x \leq \cdots \leq d_k^x$ of distances from x to the different elements of S. Consider the minimal subtree T_i of T containing x and the elements $s \in S$ with $d(x,s) \leq d_i^x$; for every integer ρ with $d_{i-1}^x \leq \rho \leq d_i^x$ denote by $E(x,i,\rho)$ the set of vertices y of the minimal subtree T_i which satisfy $d(x,y) \leq \rho$. Tamir [1985] showed that the hypergraph $(E(x,i,\rho)/x,i,\rho)$ is totally balanced.

If $S = X$ we obtain thus the hypergraph of neighbourhoods (Example 3). If $S = \{x_1\}$ we obtain the hypergraph of paths of an arborescence rooted at x_1.

Example 5. Composition of two totally balanced hypergraphs (Lubiw [1985]).

Given two hypergraphs $H = (E_1,E_2,...,E_m)$ and $H' = (F_1,F_2,...,F_n)$ on a set X, the composition hypergraph $H_{H'}$ is a hypergraph whose vertices f_i represent respectively the edges $F_i \in H'$ and whose edges are the sets $\bar{E}_j = \{f_i / F_i \cap E_j \neq \emptyset\}$. In order that $H_{\bar{H}}$ be a hypergraph on H' we suppose further that each F_i meets at least one E_j and each E_j meets at least one F_i.

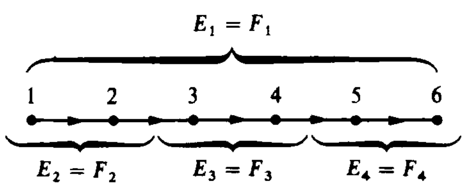


Figure 11

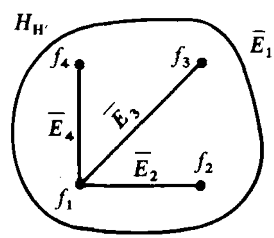


Figure 12

For example, consider the arborescence T of Figure 11 with $H = (E_1,E_2,E_3,E_4)$ and $H' = (F_1,F_2,F_3,F_4)$. Then H is the hypergraph represented in Figure 12.

Lubiw [1985] showed that if H and H' are both totally balanced then their composition hypergraph $H_{H'}$ is also totally balanced.

(Note that as in Figure 12, $H_{H'}$ need not be unimodular, even if H and H' are unimodular).

This theorem generalises a result of Frank [1977] who showed that if H and H' are two hypergraphs of paths of an arboresence then $H_{H'}$ is totally balanced; it also generalises a result of Tamir [1983] who showed that if H and H' are two hypergraphs of neighbourhoods, then $H_{H'}$ is totally balanced.

Theorem 7. *A hypergraph is balanced if and only if its induced subhypergraphs are 2-colourable.*

Proof.

1. To show that the condition is necessary it is enough to show that a balanced hypergraph is 2-colourable.

Indeed, otherwise, there exists a balanced hypergraph H of minimum order with $\chi(H) \geq 3$. For each vertex x_0, the subhypergraph induced by $X-\{x_0\}$ has a 2-colouring (S_0,S_0'), since H is minimal. As H is not 2-colourable, this implies that x_0 appears in two edges of H of cardinality 2, say $[x_0,y]$ and $[x_0,y']$, with $y \in S_0$, $y' \in S_0'$. Thus the graph G formed by the edges of H of cardinality 2 satisfies $d_G(x) \geq 2$ for every $x \in X$. Since G is a balanced hypergraph, it is a bipartite graph. Let G_1 be a connected component of G (which is of order at least 3 from the above) and let x_1 be a vertex of G_1 which is not an articulation point (there must exist at least two of these since G_1 is of order ≥ 3). The subhypergraph of H induced by $X-\{x_1\}$ has a 2-colouring, say (S_1,S_1'). Then x_1 can be coloured in such a way that no edge of G_1 is monochromatic. Thus every edge of H contains two colours if it has more than two elements, and contains also two colours if it has two elements: this contradicts $\chi(H) \neq 2$.

Observe that the existence of a 2-colouring of H also follows from the difficult theorem of Fournier-Las Vergnas (Theorem 1).

2. We shall show that if for every $A \subset X$ the subhypergraph H_A is 2-colourable, then H is balanced. Indeed, otherwise there exists an odd cycle $(a_1,E_1,a_2,E_2,\ldots,a_{2k+1},E_{2k+1},a_1)$ where no edge contains three of the a_i's. The set $A = \{a_1,a_2,\ldots,a_{2k+1}\}$ induces a subhypergraph H_A which contains the edges of the graph C_{2k+1}, and consequently H_A is not 2-colourable, a contradiction.

Corollary. *A hypergraph of rank ≤ 3 is unimodular if and only if it is balanced.*

Proof. If $r(H) \leq 3$ and if H is balanced, then there exists a 2-colouring of H, and this 2-colouring is necessarily equitable. The same is true for every subhypergraph of

H. Then, from Theorem 3, H is unimodular.

Theorem 8. *A balanced hypergraph H has a good k-colouring for every $k \geq 2$.*

Proof. Let H be a balanced hypergraph on X. For $k = 2$, the statement is proved by Theorem 7. For $k > 2$, consider a k-partition $(S_1, S_2, ..., S_k)$ of X; for each $E \in H$, denote by $k(E)$ the number of classes of this partition which meet E. If every edge $E \in H$ satisfies $k(E) = \min\{|E|, k\}$, the partition is a good k-colouring of H. Suppose that there exists an edge E_0 with $k(E_0) < \min\{|E_0|, k\}$. Since $k(E_0) < |E_0|$ there exists an index p such that $|S_p \cap E_0| \geq 2$. Since $k(E_0) < k$ there exists an index q such that $|S_q \cap E_0| = 0$.

The subhypergraph of H induced by $S_p \cup S_q$ is balanced (Proposition 1): thus it admits a 2-colouring $(\bar{S}_p, \bar{S}_q)$. Set $\bar{S}_i = S_i$ for $i \neq p, q$. Then $(\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_k)$ is also a k-partition of X, and the number $\bar{k}(E)$ of classes of this partition which meet an edge E satisfies

$$\bar{k}(E) \geq k(E) \qquad (E \in H)$$
$$\bar{k}(E_0) = k(E_0) + 1.$$

This transformation of the k-partition allows us to reduce $\min\{|E|, k\} - k(E)$ for each $E \in H$, and repeating as often as necessary, we obtain a good k-colouring of H.

Corollary 1. *A balanced hypergraph has the coloured edge property.*

Indeed, the dual hypergraph H^* of a balanced hypergraph H is of rank $r(H^*) = \Delta = \Delta(H)$. Setting $k = \Delta$ in Theorem 8 we obtain a strong colouring of the edges of H in Δ colours. Thus $q(H) = \Delta(H)$.

Applied to bipartite multigraphs (cf. *Graphs*, Chapter 12, Theorem 2), this statement gives Kőnig's theorem on edge colouring.

Corollary 2. *A balanced hypergraph H contains $k = \min_{E \in H} |E|$ pairwise disjoint transversal sets.*

It is sufficient to apply Theorem 8 with $k = \min_{E \in H} |E|$.

Applied to the dual of a bipartite graph, this gives Gupta's theorem [1978].

Corollary 3 (Las Vergnas). *Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph; denote by k_0 the least integer greater than or equal to*

$$\min_J \frac{1}{|J|} \left| \bigcup_{j \in J} E_j \right|$$

this minimum being taken over the non-empty subsets J of $\{1,2,\ldots,m\}$. Then H has a good k-colouring for every $k \leq k_0$.

Indeed, by definition of k_0, for every $J \neq \emptyset$,

$$|J|(k_0 - 1) < \left| \bigcup_{j \in J} E_j \right|$$

Let $k < k_0$; the condition in the generalisation of Theorem 3 is satisfied for k. Hence there exists a k-uniform hypergraph $H' = (E'_1, E'_2, \ldots, E'_m)$ without cycles such that $E'_i \subset E_i$ for every i. As H' is also strongly balanced, Theorem 8 shows that there exists a good k-colouring of H', which is also a good k-colouring of H. *Q.E.D.*

Theorem 9 (Berge, Las Vergnas [1970]). *A hypergraph is balanced if and only if every partial subhypergraph has the König property.*

Proof.

1. If $\nu(H'_A) = \tau(H'_A)$ for every $H' \subset H$ and $A \subset X$, then H is balanced, since otherwise there exists an H'_A isomorphic to an odd cycle C_{2k+1}; as $\nu(C_{2k+1}) = k$ and $\tau(C_{1k+1}) = k+1$, a contradiction follows.

2. If H is balanced, H'_A is also balanced. Thus it suffices to show that a balanced hypergraph satisfies $\nu(H) = \tau(H)$.

Set $\tau(H) = t$. Consider a partial hypergraph H' with $\tau(H') = t$, and such that H' is minimal with this property. We shall show that H' consists of pairwise disjoint edges, which implies

$$\nu(H) \geq \nu(H') = \tau(H') = \tau(H) \geq \nu(H);$$

consequently $\nu(H) = \tau(H)$, and the proof will be complete.

Suppose (to prove by contradiction) that two edges E'_1, E'_2 of H' satisfy $E'_1 \cap E'_2 \neq \emptyset$; let $x_0 \in E_1 \cap E_2$. There exists a transversal T_1 of $H' - E'_1$ with $|T_1| = t-1$, and there exists a transversal T_2 of $H' - E'_2$ with $|T_2| = t-1$. Let $Q = T_1 \cap T_2$, $R_i = T_i - Q$, $S = R_1 \cup R_2 \cup \{x_0\}$. The subhypergraph H'_S is balanced and thus has a 2-colouring (S_1,S_2). One of the colour classes, say S_1, satisfies $|S_1| \leq |R_1|$, since $|S| = 2|R_1|+1$. Observe that E_1' meets S in at least two points (one of them being x_0), so E_1' meets S_1, and thus meets $S_1 \cup Q$. Similarly E_2' meets $S_1 \cup Q$ (cf. Figure 13).

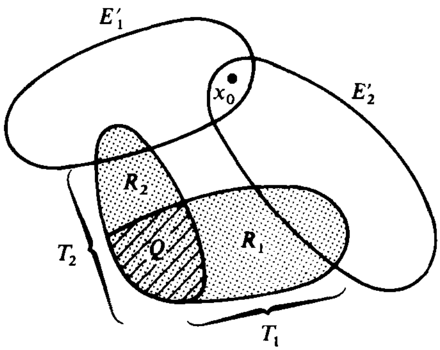


Figure 13

For $i \neq 1,2$, either the edge E_i' of H' meets Q or it meets $R_1 \cup R_2$ in at least two points, in which case E_i' meets S_1. Thus $S_1 \cup Q$ is a transversal of H', which implies that

$$\tau(H) \leq |S_1 \cup Q| \leq |R_1| + |Q| = t-1$$

A contradiction follows.

(This new proof is due to Lovász).

Corollary 1. *Every balanced hypergraph has the Helly property and is conformal.*

Proof. Let H be a balanced hypergraph and let $H' \subset H$ be an intersecting family. Since H is balanced, Theorem 9 shows that $\tau(H') = \nu(H') = 1$, and so there exists a vertex common to all the edges of H'. Thus H has the Helly property. Since the dual of a balanced hypergraph is balanced, we deduce that H is conformal.

Corollary 2. *A hypergraph H with m edges and n vertices is balanced if and only if for every* $\mathbf{c} \in \{1,+\infty\}^n$ *and each* $\mathbf{d} \in \mathbb{N}^m$, *we have*

$$N\text{-}\max_{y \in Q(c)} \langle \mathbf{d},\mathbf{y} \rangle = N\text{-}\min_{t \in P(d)} \langle \mathbf{c},\mathbf{t} \rangle .$$

Proof. 1. Let $H = (E_1,E_2,...,E_m)$ be a hypergraph for which this equality holds; consider a partial subhypergraph H'_A, and let

$$c_i = 1 \quad \text{if } x_i \in A;$$

$$c_i = +\infty \quad \text{if } x_i \notin A;$$

$$d_j = 1 \quad \text{if } E_j \in H';$$

$$d_j = 0 \quad \text{if } E_j \notin H'.$$

For $\mathbf{c} = (c_1,c_2,...,c_n)$ and $\mathbf{d} = (d_1,d_2,...,d_m)$, we have:

$$\nu(H'_A) = N\text{-}\max_{y \in Q(c)} \langle \mathbf{d},\mathbf{y} \rangle = N\text{-}\min_{t \in P(d)} \langle \mathbf{c},\mathbf{t} \rangle = \tau(H'_A).$$

Thus, from Theorem 9, H is balanced.

2. Let H be a balanced hypergraph; it suffices to show that for $\mathbf{d} \in \mathbb{N}^m$, we have:

$$N\text{-}\max_{y \in Q(1)} \langle \mathbf{d},\mathbf{y} \rangle = N\text{-}\min_{t \in P(d)} \langle \mathbf{1},\mathbf{t} \rangle$$

If we associate with each edge E_j of H an integer $d_j \geq 0$ called its "weight" then it is enough to show that the maximum weight of a matching is equal to the minimum value of a **d**-transversal. For an integer $\lambda > 0$, an edge $E = \{x_1,x_2,...,x_r\}$ will be *duplicated* λ times if we replace each $x_i \in E$ by a set $X_i = \{x_i^1,x_i^2,...,x_i^\lambda\}$ of λ additional vertices, and the edge E by λ new edges $E^1 = \{x_1^1,x_2^1,...,x_r^1\}$, $E^2 = \{x_1^2,x_2^2,...,x_r^2\}$ etc. We say that the edge E is duplicated 0 times if we remove the edge E from H.

In each case, the hypergraph so obtained is balanced. For $\mathbf{d} = (d_1,d_2,...,d_m)$, denote by $H^{[d]}$ the hypergraph obtained from H by duplicating the edge E_1 d_1 times, the edge E_2 d_2 times, etc.

It is easily seen that

$$N\text{-}\max_{y \in Q(1)} \langle \mathbf{d},\mathbf{y} \rangle = \nu(H^{[d]})$$

$$N\text{-}\min_{t \in P(d)} \langle \mathbf{1},\mathbf{t} \rangle = \tau(H^{[d]})$$

Since $H^{[d]}$ is a balanced hypergraph, these two coefficients are equal, which achieves the proof.

Theorem 10 (Fulkerson, Hoffman, Oppenheim [1974]). *Let H be a balanced hypergraph with m edges and n vertices. For each $\mathbf{c} \in \mathbb{N}^n$, we have*

$$N\text{-}\max_{y \in Q(c)} \langle \mathbf{1}, \mathbf{y} \rangle = N\text{-}\min_{t \in P(1)} \langle \mathbf{c}, \mathbf{t} \rangle$$

(*) **Proof.** We shall assume some knowledge of the theory of linear programming.

1. We shall show that the program:

(1) minimize $\langle \mathbf{c}, \mathbf{t} \rangle$ for $\mathbf{t} \in P(\mathbf{1})$

has an integer solution.

From Corollary 2 of Theorem 9, the maximum of $\langle \mathbf{d}, \mathbf{y} \rangle$ for those $\mathbf{y}$ in the polyhedron $Q = \{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ \mathbf{y} \geq \mathbf{0},\ A\mathbf{y} \leq \mathbf{1}\}$ is attained at a point $\mathbf{y}_0$ with integer coordinates, indeed 0,1 coordinates since $A\mathbf{y}_0 \leq \mathbf{1}$.

Since this is true for all $\mathbf{d} \in \mathbb{N}^m$, it is easy to see that all the extreme points of Q have coordinates 0,1 (cf. for example Lemma 1 of §6). The hyperplane $\{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ A\mathbf{y} = \mathbf{1}\}$ being a supporting hyperplane of the convex polyhedron Q, all the extreme points of the polyhedron $\overline{Q} = \{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ \mathbf{y} \geq \mathbf{0},\ A\mathbf{y} = \mathbf{1}\}$ have 0,1 coordinates. Let $\mathbf{z}$ be an extreme point of the polyhedron $\{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m,\ \mathbf{y} \geq \mathbf{0},\ A\mathbf{y} \geq \mathbf{1}\}$; this is also an extreme point of the polyhedron obtained by eliminating the inequalities of $A\mathbf{y} \geq \mathbf{1}$ which are strict. Thus $\mathbf{z}$ has integer coordinates. Applying this result to the dual H^* of H, which is also balanced, we see that the extreme points of the polyhedron

$$P(\mathbf{1}) = \{\mathbf{t}/\mathbf{t} \in \mathbb{R}^n,\ \mathbf{t} \geq \mathbf{0},\ A^*\mathbf{t} \geq \mathbf{1}\}$$

have integer coordinates, whence the result.

2. We shall show that the program:

(2) maximise $\langle \mathbf{1}, \mathbf{y} \rangle$ for $\mathbf{y} \in Q(\mathbf{c})$

has a solution with integer coordinates. (This result, combined with the result of part 1 above immediately implies the equality in the statement of Theorem 10.) We shall argue by a double induction, on $\Sigma c_i = \lambda$ and on m; the result is clear if $\lambda = 1$ or if $m = 1$.

Let $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ be a solution to the program (2) with fractional coordinates. If $z_j = 0$, it suffices to apply the induction hypothesis with $m-1$ to the submatrix of A obtained by eliminating the jth column to show that the program (2) has a solution with integer coordinates. Thus we may suppose that $z_j > 0$ for every j.

By virtue of part 1 and the duality theorem, we know that $\langle \mathbf{1}, \mathbf{z} \rangle = k$ is an integer. Suppose that the ith row vector a^i of the matrix satisfies $\langle \mathbf{z}, \mathbf{a}^i \rangle < c_i$. If $\langle \mathbf{z}, \mathbf{a}^i \rangle \leq c_i - 1$ we may apply the induction hypothesis with $\lambda-1$ to show the existence of an integer solution $\bar{z}$ of the programme (2) with $\langle \mathbf{1}, \bar{\mathbf{z}} \rangle = \langle \mathbf{1}, \mathbf{z} \rangle = k$.

Hence we may suppose that $\langle \mathbf{z},\mathbf{a}^i \rangle = c_i - 1 + \epsilon$ with $0 < \epsilon < 1$. Clearly there exists a vector $\bar{\mathbf{z}} \in P(c_1,c_2,...,c_i-1,...,c_n)$ with $\bar{\mathbf{z}} \leq \mathbf{z}$, $\langle \mathbf{1},\bar{\mathbf{z}} \rangle = k - \epsilon$. By the induction hypothesis with $\lambda-1$, there exists a vector $\bar{\bar{\mathbf{z}}}$ with integer coordinates such that

$$\bar{\bar{\mathbf{z}}} \geq 0,\ A\bar{\bar{\mathbf{z}}} \leq (c_1,...,c_i-1,...,c_n) \leq \mathbf{c},\ \ \langle \mathbf{1},\bar{\bar{\mathbf{z}}} \rangle \geq k - \epsilon.$$

Hence $\langle \mathbf{1},\bar{\bar{\mathbf{z}}} \rangle = k$ and the demonstration is done.

Thus we may suppose $\langle \mathbf{z},\mathbf{a}^i \rangle = c_i$ for every i, and $z_j > 0$ for every j. By virtue of the principle of complementary slackness, each optimal solution $\bar{\mathbf{x}}$ of the dual program:

(3) minimise $\langle \mathbf{c},\mathbf{x} \rangle$ for $\mathbf{x} \in P(1)$

satisfies $A^*\bar{\mathbf{x}} = \mathbf{1}$, $\bar{\mathbf{x}} \geq \mathbf{0}$, $\langle \mathbf{c},\bar{\mathbf{x}} \rangle = k$. Hence $\bar{\mathbf{z}}$ and $\bar{\mathbf{x}}$ are optimal solutions respectively of the dual programs:

(4) minimise $\langle \mathbf{1},\mathbf{y} \rangle$ for $\mathbf{y} \in Q(\mathbf{c})$;

(5) maximise $\langle \mathbf{c},\mathbf{x} \rangle$ for $\mathbf{x} \in \{\mathbf{x}/\mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \geq \mathbf{0},\ A^*\mathbf{x} \leq \mathbf{1}\}$.

Furthermore, $\langle \mathbf{1},\mathbf{z} \rangle = \langle \mathbf{c},\bar{\mathbf{x}} \rangle$.

As we have seen in part 1, there exists a vector $\bar{\mathbf{z}}$ with integer coordinates such that $\bar{\mathbf{z}} \geq \mathbf{0}$, $A\bar{\mathbf{z}} \geq \mathbf{c}$, $\langle \mathbf{1},\bar{\mathbf{z}} \rangle = k$. If $A\bar{\mathbf{z}} = \mathbf{c}$ the demonstration is done. Suppose therefore that $\langle \bar{\mathbf{z}},\mathbf{a}^i \rangle > c_i$ for an $i \leq n$. Since $z_j > 0$ for every j, there exists an ϵ with $0 < \epsilon < 1$ such that $z_j > (1-\epsilon)\bar{z}_j$ for every j. Set

$$\mathbf{w} = \frac{1}{\epsilon}[\mathbf{z}-(1-\epsilon)\bar{\mathbf{z}}]$$

Then $\mathbf{z} = (1-\epsilon)\bar{\mathbf{z}} + \epsilon\mathbf{w}$, $\mathbf{w} \geq \mathbf{0}$, $\langle \mathbf{1},\mathbf{w} \rangle = k$. As $A\mathbf{z} = \mathbf{c}$ and $A\bar{\mathbf{z}} \geq \mathbf{c}$, we deduce that $A\mathbf{w} \leq \mathbf{c}$. Further, since there exists an i such that $\langle \mathbf{z},\mathbf{a}^i \rangle > c_i$ we have $\langle \mathbf{w},\mathbf{a}^i \rangle < c_i$. Thus $\mathbf{w}$ is a solution of (2) with $\langle \mathbf{w},\mathbf{a}^i \rangle < c_i$ for some i. Applying the induction hypothesis once more, to show the existence of a solution $\bar{\mathbf{w}}$ of the program (2) with integer coordinates, we complete the proof.

Remark. Let H be a balanced hypergraph. From Theorem 10 and Corollary 2 of Theorem 9, we see that many values of $\mathbf{c}$ and $\mathbf{d}$ satisfy

$$N\!-\!\max_{y \in Q(c)} \langle \mathbf{d},\mathbf{y} \rangle = N\!-\!\min_{t \in P(d)} \langle \mathbf{c},\mathbf{t} \rangle$$

Nonetheless, this equality is not satisfied for all $\mathbf{c}$ and $\mathbf{d}$ for a balanced hypergraph (as it is for unimodular hypergraphs).

Consider for example the balanced hypergraph of Figure 12, with $\mathbf{c} = (3,2,2,2)$ and $\mathbf{d} = (2,1,1,1)$. The vector $\mathbf{t} = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is a fractional $\mathbf{d}$-transversal since

$$\sum_{f_i \in E_j} t_i \geq d_j \quad (j = 1,2,3,4).$$

The vector $\mathbf{y} = (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is a fractional $\mathbf{c}$-matching since

$$\sum_{j/f_i \in E_j} y_j \leq c_i \quad (i = 1,2,3,4).$$

We have $\langle \mathbf{d},\mathbf{y} \rangle = \langle \mathbf{c},\mathbf{t} \rangle = \frac{9}{2}$ and the minimax equality does not hold, since

$$\underset{y \in Q(c)}{N\text{-}max} \langle \mathbf{d},\mathbf{y} \rangle < \frac{9}{2} < \underset{t \in P(d)}{N\text{-}min} \langle \mathbf{1},\mathbf{t} \rangle.$$

Application. Location problems.

Given a tree T on a set $X = \{x_1,x_2,...,x_n\}$, we may interpret the vertex x_i as a possible centre capable of distributing consumer goods to all vertices x such that $d(x,x_i) \leq \rho_i$, where ρ_i is a given integer ≥ 0. Further, each vertex x_i has an annual cost c_i of maintenance of a distribution centre. The problem consists of choosing a set of distribution centres, capable of serving all the clients, for which the total cost is as small as possible. If H is the hypergraph whose edges are the $T_i = \{x / d(x_i,x) \leq \rho_i\}$ then we require a minimum cost cover of H, that is to say for the dual H^* a minimum cost transversal. From Theorem 10, we have

$$\underset{y \in Q(c)}{N\text{-}max} \langle \mathbf{1},y \rangle = \underset{t \in P(1)}{N\text{-}min} \langle c,t \rangle.$$

Thus the minimum cost of a cover of H is equal to the maximum cardinality of a family of vertices of H which has at most c_j representatives in the edge E_j for $j = 1,2,...,m$. Polynomial time algorithms to determine optimal locations are due to Tamir [1980], Kolen [1982], Farber [1984], Lubiw [1984].

To recognize whether a hypergraph is totally unimodular and to determine an optimal d-value c-matching it is useful to consider a particular order relation on the set of d-dimensional vectors (Lubiw [1974]). This relation, which we shall call reverse lexicographic order and denote by $\lesssim$, is defined by

$$(r_1,r_2,...) \lesssim (s_1,s_2,...)$$

if the largest index k such that $r_k \neq s_k$ satisfies $r_k < s_k$.

Lemma 1. *In every 0,1 matrix the rows and columns can be simultaneously arranged in reverse lexicographic order.*

Proof. Consider a 0,1-matrix $A = ((a_j^i))$ with m columns, n rows. Consider the vector $\mathbf{d}_A = (d_2, d_3, \ldots, d_{m+n})$ where $d_k = \sum_{i+j=k} a_j^i$.

If for two indices j_1, j_2 with $j_1 < j_2$ the column vectors corresponding to $\mathbf{a}_{j_1}$ and $\mathbf{a}_{j_2}$ are in the wrong order, i.e. $\mathbf{a}_{j_2} \lesssim \mathbf{a}_{j_1}$, the matrix A' obtained by permuting the columns j_1 and j_2 satisfies $\mathbf{d}_A \lesssim \mathbf{d}_{A'}$. Then, taking the permutation of rows and columns of the initial matrix A which maximises $\mathbf{d}_A$ we obtain a new matrix satisfying the required conditions.

Lemma 2. *If a matrix $A = ((a_j^i))$ has its rows and columns arranged in reverse lexicographic order, and if A contains a submatrix equal to*

$$\begin{pmatrix} a_{j_1}^{i_1} & a_{j_2}^{i_2} \\ a_{j_1}^{i_2} & a_{j_2}^{i_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = B$$

with $i_1 < i_2$, $j_1 < j_2$, then in the hypergraph H the vertices x_{i_1}, x_{i_2} and the edges E_{j_1}, E_{j_2} appear in a cycle of length ≥ 3 with no edge containing 3 vertices of the cycle.

(It is easy to show with an inductive construction that the given submatrix occurs with an unbalanced cycle).

Theorem 1 (Hoffman, Sakarovitch, Kolen [1985], Lubiw [1985]). *Let $A = ((a_j^i))$ be the incidence matrix of a hypergraph H. The following conditions are equivalent:*

(i) *the matrix A with its rows and columns arranged in reverse lexicographic order contains no such submatrix B;*

(ii) *it is possible to place the rows and columns of A in an order such that A contains no submatrix B;*

(iii) *the hypergraph H is totally balanced.*

Proof.

(i) implies (ii). Clear.

(ii) implies (iii). Indeed, if H is not totally balanced, there exists a cycle

$$(x_{i_1},E_{j_1},x_{i_2},...,E_{j_k},x_{i_1})$$

with $k \geq 3$ where each edge contains exactly two vertices of the cycle. The submatrix of A generated by the rows $i_1,i_2,...$ and the columns $j_1,j_2,...$ contains exactly two 1's in each row and in each column (whatever the order of their indices); the two columns which have a 1 in the top row, together with the top row and the row which has a 1 under the first 1, gives the matrix B: this contradicts (ii).

(iii) implies (i). From Lemma 2.

Remark 1. Condition (i) provides an effective algorithm to determine whether a hypergraph H is totally balanced (Lubiw [1985], Hoffman, Sakarovitch, Kolen [1985]). This algorithm appears to perform better than other polynomial algorithms which had previously been proposed (Fagin [1983], Farber [1983], Anstee and Farber [1984]). Observe that this recognition problem is of practical interest in the study of database schemes (Fagin [1983]).

Remark 2. Hoffman, Sakarovitch and Kolen called a 0,1-matrix *greedy* if it satisfies condition (ii), and they showed that a maximum **d**-valued **c**-matching may be obtained by a greedy algorithm for every $\mathbf{d} \in \mathbb{N}^m$ and every $\mathbf{c} \in \mathbb{N}^n$ if and only if the matrix A is greedy. This is thus a characteristic property of totally balanced matrices. Moreover, it indicates how to obtain a minimum **c**-valued **d**-transversal in polynomial time when $\mathbf{d} \in \mathbb{N}^m$, $\mathbf{c} \in \mathbb{N}^n$.

Remark 3. Farber [1982], [1985] independently obtained logically equivalent results by a different approach relating to Graph Theory. Recall that a graph G is said to be *triangulated* if every cycle of length ≥ 4 has a chord (cf. *Graphs*, Chap. 16 §3). A *sun* of G is a subgraph induced by a set $S = \{a_1,a_2,...,a_k,b_1,b_2,\dots,b_k\}$ with $k \geq 3$ which is the union of the complete graph on $\{a_1,a_2,...,a_k\}$ and the cycle $(a_1,b_1,a_2,b_2,...,b_k,a_1)$. Farber showed that for a graph G on X, the following conditions are equivalent:

(i) G is triangulated and sun-free;

(ii) each subgraph G' of G contains a vertex x such that the family $(\{y \cup \Gamma_{G'}(y) / y \in \Gamma_{G'}(x)\})$ is totally ordered by inclusion;

(iii) the vertices x_i can be indexed in such a way that the adjacency matrix of G contains no submatrix B;

(iv) the sets $\{x\} \cup \Gamma_G(x)$ form a totally balanced hypergraph on X;

(v) the maximal cliques of G form a totally balanced hypergraph on X.

4. Arboreal Hypergraphs

A hypergraph H is *arboreal* if:

(i) H satisfies the Helly property;

(ii) each cycle of length ≥ 3 contains three edges having a non-empty intersection.

A hypergraph H is *co-arboreal* if it is the dual of an arboreal hypergraph, that is to say if:

(i′) H is conformal;

(ii′) every cycle of length ≥ 3 has three vertices contained in the same edge of H.

Example. A totally balanced hypergraph is both arboreal and co-arboreal. Indeed, from Corollary 1 of Theorem 9, such a hypergraph H satisfies (i) and (i′). Further it is clear that H satisfies (ii) and (ii′). In fact, a hypergraph is totally balanced if and only if all of its induced subhypergraphs are arboreal.

Observe that the hypergraph of Figure 14, whose edges are *abd*, *bcd*, *acd* is arboreal, but it is not totally balanced, since *abc* are the three vertices of a cycle, and no edge contains the three.

Theorem 2. *A simple hypergraph is the family of maximal cliques of a triangulated graph if and only if it is co-arboreal.*

Proof. 1. Let H be a simple co-arboreal hypergraph. As H is conformal from (i′), it is the hypergraph of maximal cliques of $G = [H]_2$, the 2-section of H. Further, G is triangulated as otherwise it would contain a cycle of length ≥ 4 without chords, which corresponds in H to a cycle with no edge containing three vertices of the cycle: a contradiction with (ii′).

2. Let H be the hypergraph of maximal cliques of a triangulated graph G. Then H is conformal and satisfies (i′). Further, a cycle $\mu = [a,b,...]$ of G has three vertices contained in the same edge of H if its length is 3 (since H is conformal). If the length of μ is ≥ 4, the partial subgraph $G_\mu - [a,b]$, which is connected, has a shorter path between a and b of the form $[a,x,b]$ (as G_μ is triangulated) which shows that the three

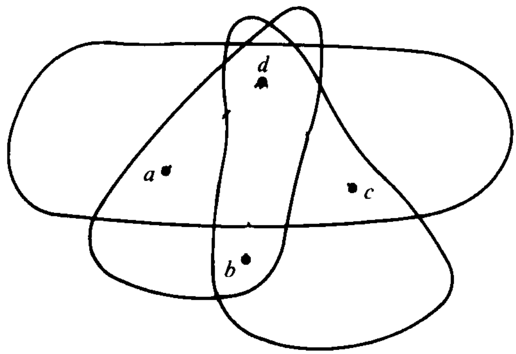


Figure 14. *Arboreal hypergraph (not totally balanced).*

vertices a,b,x of μ are contained in the same edge of H. Thus (ii′) holds. Thus we have shown that H is co-arboreal.

Corollary. *A hypergraph H is arboreal if and only if H satisfies the Helly property and the representative graph $L(H)$ is triangulated.*

Indeed, we have seen (Proposition 1, §8, Chap. 1) that if a hypergraph H satisfies the Helly property, a graph G is the representative of H if and only if H^* is the hypergraph of the maximal cliques of G (with, perhaps, other cliques of G). From Theorem 12, this graph G is triangulated if and only if H^* is co-arboreal, i.e. H is arboreal.

Lemma. *Let H be an arboreal hypergraph without loops; there exists a vertex x_0 such that all the edges of H containing x_0, have a common vertex $y_0 \neq x_0$.*

Proof. Let $(x_1,E_1,x_2,\ldots,x_q,E_q,x_{q+1},\ldots,E_p,x_{p+1})$ be a *path* of H with $E_i \cap E_j = \emptyset$ if $|i-j| > 1$, $x_1 \notin E_2$ and $x_{p+1} \notin E_{p-1}$. Suppose that it is maximal in length and set $x_0 = x_{p+1}$. By virtue of the maximality of this path, an edge $E_\lambda \in H$ with $x_0 \in E_\lambda$, $|E_\lambda| \neq 1$, $E_\lambda \neq E_p$ satisfies $E_\lambda \cap E_q \neq \emptyset$ for some $q \leq p-1$.

Assume that q is the largest possible index defined in this way. The edges $E_q,E_{q+1},\ldots,E_p,E_\lambda$ define a cycle, and as H is arboreal, we must have

$$E_\lambda \cap (E_q \cap E_{q+1}) \neq \emptyset.$$

From the maximality, we have $q = p-1$, whence $E_\lambda \cap E_{p-1} \cap E_p \neq \emptyset$.

As this is true for every edge E_λ with $x_0 \in E_\lambda$, $|E_\lambda| \neq 1$, the family formed by E_{p-1}, E_p and all the E_λ is intersecting; then by the Helly property these edges have a common element y_0. Further we have $y_0 \neq x_0$ since $x_0 \notin E_{p-1}$.

Q.E.D.

Theorem 13 (Duchet [1978], Flament [1978], Slater [1978]). *A hypergraph H on X is arboreal if and only if there exists a tree T on X such that the edges of H induce subtrees of T.*

Proof.

1. Let H be a hypergraph of subtrees of T. We know from Theorem 10, Chap. 1, that H satisfies the Helly property. Further, a cycle $(x_1, E_1, x_2, E_2, \ldots, x_k)$ of H of length ≥ 3 having no three edges with a non-empty intersection determines a sequence $\mu[x_1,x_2], \mu[x_2,x_3], \ldots$ of paths of T with $x_j \notin \mu[x_i, x_{i+1}]$ if $j > i+1$, which contradicts $x_k = x_1$. Consequently, the hypergraph H is indeed arboreal.

2. Let H be an arboreal hypergraph on X. We shall demonstate the existence of a tree T with the required properties by induction on $|X|$.

Let x_0, y_0 be vertices of H defined as in the lemma; the subhypergraph $\overline{H}$ induced by $\overline{X} = X - \{x_0\}$ is also arboreal since it satisfies (i) and (ii); thus by the induction hypothesis there exists a tree $\overline{T}$ on $\overline{X}$ satisfying the desired property for $\overline{H}$. Clearly the tree $T = \overline{T} + [x_0, y_0]$ satisfies the desired property.

Q.E.D.

(This new proof is due to Duchet).

Application. If we represent species of animals at present in existence by the vertices of a hypergraph, with each edge being a set of species presenting a common hereditary characteristic, the theory of evolution says that this hypergraph is arboreal.

Observe that for the arboreal hypergraph of Figure 14, the corresponding tree T is uniquely determined. In general, a hypergraph H may have many corresponding trees; for a complete description of these trees, cf. Duchet [1985].

To determine whether a given hypergraph is arboreal we shall use an extension of the concept of the "cyclomatic number of a graph" due to Acharya and Las Vergnas.

Given a hypergraph $H = (E_1, E_2, \dots, E_m)$ on X, its representative graph $L(H)$ will be "weighted" by associating with each edge $u = [e_i, e_j]$ the integer $w(u) = |E_i \cap E_j|$ which we call its "weight". If F is a partial graph of $L(H)$ without cycles ("forest" of $L(H)$), the weight of F is defined to be $w(F) = \sum_{u \in F} w(u)$.

Finally, we define the *cyclomatic number* of the hypergraph H to be the integer

$$\mu(H) = \sum_{j=1}^{m} |E_j| - |X| - w_H,$$

where w_H is the maximum weight of a forest $F \subset L(H)$.

For example, the reader may verify that the hypergraph H of Figure 15 contains a forest of maximum weight 5 (in fact, F is a tree since $L(H)$ is connected); the cyclomatic number of H is then $\mu(H) = 12 - 6 - 5 = 1$.

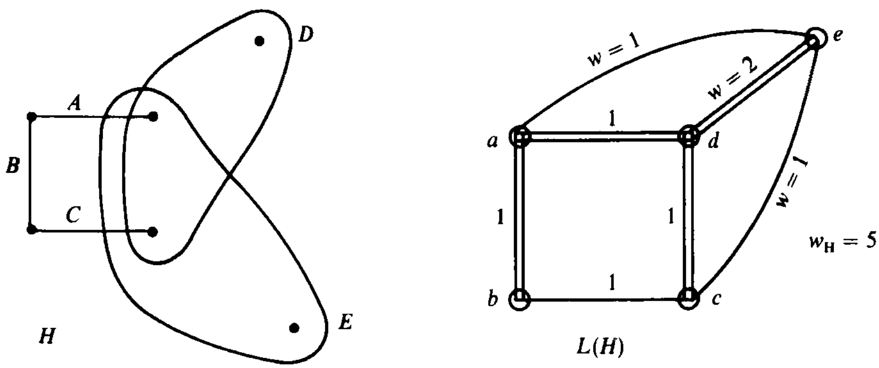


Figure 15. *A balanced hypergraph, not co-arboreal, and its representative weighted graph.*

The determination of the cyclomatic number $\mu(H)$ is an easy problem, as it reduces to the classical problem of the determination of a maximum weight tree in a graph with (positive) weighted edges; the complexity of various algorithms (e.g. Kruskal, Solin, Hell, etc.) has been studied. Recall, for example, Kruskal's greedy algorithm: form a forest edge by edge, each time taking the edge of greatest weight which will not create a cycle with the edges already chosen.

Remark. If H is a linear hypergraph of order n with m edges and p connected components, then each edge of $L(H)$ is of weight 1 and the maximum weight forest F has weight $w(F) = n(F) - p(F) = m-p$. Then

$$\mu(H) = \Sigma |E_j| - n - m + p.$$

In particular, if H is a simple graph, we obtain

$$\mu(H) = 2m - n - m + p = m - n + p.$$

We thus recover the expression for the cyclomatic number of a simple graph.

If H has only one edge E_1, then

$$\mu(H) = |E_1| - |E_1| = 0.$$

If H has just two edges E_1 and E_2, then

$$\mu(H) = |E_1| + |E_2| - |E_1 \cup E_2| - |E_1 \cap E_2| = 0.$$

If H has more than two edges we have $\mu(H) \geq 0$, as we see immediately (by induction on the number of edges) with the following proposition:

Proposition 1. *Let H be a hypergraph with more than two edges. Then there exists an edge E_1 of H such that $\mu(H) \geq \mu(H-E_1)$; further there exists an edge E_2 such that*

$$\mu(H) - \mu(H-E_1) \geq |E_1| - |E_1 \cap E_2| - |E_1 - \bigcup_{j \neq 1} E_j| \geq 0.$$

Proof. Let e_1 be a vertex of degree 1 of the maximum weight forest $F \subset L(H)$. Let e_2 be the vertex adjacent to e_1 in F. The partial hypergraph $H' = H - E_1$ obtained by omitting the edge E_1 corresponding to e_1 satisfies

$$w_{H'} \geq w(F-[e_1,e_2]) = w_H - |E_1 \cap E_2|.$$

Hence:

$$\begin{aligned}\mu(H) - \mu(H') &= \Sigma |E_j| - |\bigcup E_j| - w_H - (|\Sigma E_j| - |E_1|) \\ &\qquad + (|\bigcup E_j| - |E_1 - \bigcup_{j \neq 1} E_j|) + w_{H'} \\ &\geq |E_1| - |E_1 - \bigcup_{j \neq 1} E_j| + w_{H'} - w_H \\ &\geq |E_1| - |E_1 - \bigcup_{j \neq 1} E_j| - |E_1 \cap E_2| \geq 0\end{aligned}$$

Q.E.D.

Theorem 14 (Acharya, Las Vergnas [1982]). *A hypergraph H satisfies $\mu(H) = 0$ if* *and only if H is co-arboreal (i.e. from the corollary to Theorem 12, if H is the hypergraph of cliques of a triangulated graph.)*

Proof.

1. Let H be a co-arboreal hypergraph on X. We shall show that $\mu(H) = 0$ by induction on $\sum_{j=1}^{m} |E_j|$.

- If $\Sigma |E_j| = 1$ the hypergraph has a single edge, which is indeed a loop, so

$$\mu(H) = \Sigma |E_j| - |X| - w_H = 1 - 1 - 0 = 0,$$

- If $\Sigma |E_j| \geq 2$, consider two cases.

Case 1: The hypergraph H has a vertex x_1 of degree 1. The subhypergraph $\overline{H}$ of H induced by $X - \{x_1\}$ satisfies

$$\mu(\overline{H}) = (\Sigma |E_j| - 1) - (n-1) - w_H = \mu(H).$$

The hypergraph $\overline{H}$ is also co-arboreal by the axioms (i′) and (ii′). Since $\sum_{\overline{E} \in \overline{H}} |\overline{E}| < \sum_{E \in H} |E|$ we have, by the induction hypothesis, $\mu(\overline{H}) = 0$, hence $\mu(H) = 0$.

Case 2: The hypergraph H has two edges E_1 and E_2 with $E_1 \subset E_2$. The partial hypergraph $H' = H - E_1$ satisfies $w_{H'} = w_H - |E_1|$ from Kruskal's algorithm, whence

$$\mu(H') = (\Sigma |E_j| - |E_1|) - n - (w_H - |E_1|) = \mu(H).$$

As H' is co-arboreal from axioms (i′) and (ii′), and as $\sum_{E' \in H'} |E'| < \sum_{E \in H} |E|$ we have

$$\mu(H') = 0$$

by the induction hypothesis, hence $\mu(H) = 0$.

We are necessarily in case 1 or case 2, since H is the hypergraph of cliques of a triangulated graph (Theorem 12) and we know that a triangulated graph has a vertex which appears in only one maximal clique (cf. *Graphs*, Chap. 16 §3). The proof is thus complete.

2. Let H be a hypergraph with $\mu(H) = 0$. We shall show by induction on the number of edges that H is co-arboreal.

We may suppose that H has at least two edges (otherwise the result is clear); from Proposition 1, there exist two edges E_1 and E_2 such that

$$0 = \mu(H) \geq \mu(H-E_1) + |E_1| - |E_1 \cap E_2| - |E_1 - \bigcup_{j \neq 1} E_j|$$
$$\geq \mu(H-E_1) + |E_1| - |E_1 \cap E_2| - |E_1 - E_2|$$
$$\geq \mu(H-E_1) \geq 0.$$

We thus have equality throughout, and in particular,

(1) $\qquad \mu(H) = \mu(H-E_1) = 0$

(2) $\qquad |E_1 - \bigcup_{j \neq 1} E_j| = |E_1 - E_2|.$

By (1) and the induction hypothesis, $H-E_1$ is the family of maximal cliques of a triangulated graph G' (plus, perhaps, other non-maximal cliques); the graph G obtained from G' by joining pairs of vertices contained in E_1 is also triangulated, because of (2). Thus the hypergraph H is co-arboreal.

Q.E.D.

Corollary 1. *A hypergraph H is arboreal if and only if $\mu(H^*) = 0$.*

The recognition of arboreal hypergraphs is thus simple, as it reduces to the problem of maximum weight trees.

Corollary 2 (Lovász' Inequality). *Let $H = (E_1, E_2, \ldots, E_m)$ be a coarboreal hypergraph. Set*

$$s = \max_{i \neq j} |E_i \cap E_j|.$$

Then we have:

(1) $\qquad \sum_{j=1}^{m} (|E_j| - s) \leq n - s.$

Indeed, as H is connected, the maximum weight forest $F \subset L(H)$ is a tree, and satisfies

$$w(F) \leq s(n(F)-1) = sm-s$$

whence

$$0 = \mu(H) = \Sigma\,|E_j| - n - w(F) \geq \Sigma\,|E_j| - n - sm + s.$$

The inequality (1) is thus satisfied.

Remark. Inequality (1) was demonstrated by Lovász [1968] in the case where H has no cycles of length ≥ 3 and where $s = 2$; it was studied by Hansen and Las Vergnas [1974] in the case where H has no cycles of length ≥ 3 and where $s \geq 2$. As has been noted by Acharya [1983] inequality (1) is satisfied in lots of other cases; for example, for the hypergraph H of Figure 15 we have $s = 2$ and

$$\Sigma(|E_j|-2) = 2 \leq n-2 = 4.$$

Thus inequality (1) is also satisfied. Zhang and Li [1983] have shown that (1) holds also if H has no odd cycles and if every cycle has two vertices contained in at least two common edges.

5. Normal Hypergraphs

We say that a hypergraph H is *normal* if every partial hypergraph H' has the coloured edge property, that is to say

$$q(H') = \Delta(H') \qquad (H' \subset H).$$

Example 1. A balanced hypergraph is normal. Indeed, we have seen that every partial hypergraph of a balanced hypergraph is also balanced, and that every balanced hypergraph has the coloured edge property (Corollary 1 to Theorem 8).

Note that the converse is not true: for example, the hypergraph H of Figure 16 is normal, but it is not balanced. In fact, it was in order to generalise results on balanced hypergraphs that Lovász [1972] introduced the concept of a normal hypergraph.

Example 2 (Shearer [1982]). The hypergraph of a simply connected polyomino is normal.

Theorem 15 (Fournier, Las Vergnas [1972]). *Every normal hypergraph is 2-colourable.*

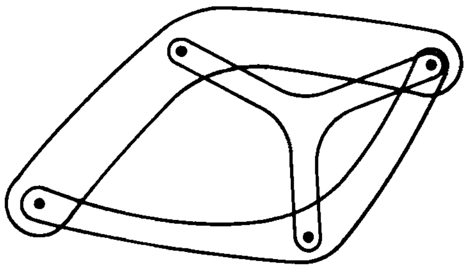


Figure 16. *A normal hypergraph (not balanced).*

Indeed, a normal hypergraph H cannot contain an odd cycle $(x_1,E_1,x_2,E_2,\dots,E_{2k+1},x_1)$ such that $H' = (E_1,E_2,\dots,E_{2k+1})$ is of maximum degree $\Delta(H') = 2$, as this would imply $q(H') \geq 3$. From Theorem 1 we deduce that $\chi(H) \leq 2$.

We shall now establish the fundamental result of this chapter, Lovász's Theorem.

As a preliminary we shall prove the following lemma:

Lemma. *Let $H = (E_1,\dots,E_m)$ be a normal hypergraph on X. If E_{m+1} is a subset of X equal to E_1, the hypergraph $H' = H + E_{m+1}$ is also normal.*

Proof. It suffices to show that $q(H') = \Delta(H')$.

Case 1: The set E_1 contains a vertex x with $d_H(x) = \Delta(H)$. In this case, $\Delta(H') = \Delta(H) + 1$, so

$$\Delta(H') \leq q(H') \leq q(H) + 1 = \Delta(H) + 1 = \Delta(H')$$

We thus have equality throughout, and $q(H') = \Delta(H')$.

Case 2: The set E_1 contains no vertex x with $d_H(x) = \Delta(H)$.

Set $\Delta(H) = \Delta$, and consider an optimal colouring of the edges of H with Δ colours; let α be the colour given to the edge E_1. Let H_α be the family of edges of H other than E_1 which receive the colour α. A vertex x with $d_H(x) = \Delta$ must necessarily appear in an edge of colour α other than E_1, so $\Delta(H-H_\alpha) = \Delta-1$. Since H is normal we have

$$q(H-H_\alpha) = \Delta(H-H_\alpha) = \Delta - 1.$$

Thus there exists a colouring of the edges of $H-H_\alpha$ with $\Delta-1$ colours, and if we add a new colour to colour the edges of $H_\alpha + E_{m+1}$ we obtain a Δ-colouring of H'. Hence

$$q(H') \leq \Delta = \Delta(H') \leq q(H').$$

Thus $q(H') = \Delta(H')$.

Theorem 16 (Lovász [1972]). *Let $H = (E_1,E_2,...,E_m)$ be a hypergraph of order n and let A be its incidence matrix with n rows, m columns. The following conditions are equivalent:*

(1) *H is normal, i.e. every partial hypergraph $H' \subset H$ has the coloured edge property;*

(2) *every extreme point of the matching polytope $Q = \{\mathbf{y}/\mathbf{y} \in \mathbb{R}^m, \mathbf{y} \geq \mathbf{0}, A\mathbf{y} \leq \mathbf{1}\}$ is a 0,1 vector;*

(3) *every extreme point of the matching polytope is integer valued;*

(4) $N\text{-}\max_{y \in Q(1)} \langle \mathbf{d},\mathbf{y} \rangle = N\text{-}\min_{t \in P(d)} \langle \mathbf{1},\mathbf{t} \rangle$ *for every* $\mathbf{d} \in \mathbb{N}^m$;

(5) *every partial hypergraph $H' \subset H$ has the König property.*

Proof.

(1) implies (2). Let $\mathbf{z}$ be an extreme point of the polyhedron Q. As $\mathbf{z}$ is the solution of a set of linear equalities with integer coefficients, each coordinate of the vector $\mathbf{z}$ is a rational number: thus there exist integers $p_1,p_2,...,p_m$ and $k \geq 0$ such that $k\mathbf{z} = (p_1,p_2,...,p_m)$.

Let H' be the hypergraph obtained from H by repeating each edge E_i p_i times. From the lemma, H' is normal. Further, for $x_i \in X$ we have

$$d_{H'}(x_i) = \sum_{j \mid x_i \in E_j} p_j = \langle \mathbf{a}^i, k\mathbf{z} \rangle = k \langle \mathbf{a}^i, \mathbf{z} \rangle \leq k.$$

Thus $q(H') = \Delta(H') \leq k$ and we may consider a k-colouring of the edges of H' with colours $1,2,...,k$. Set

$$y_j(\alpha) = 1 \text{ if a copy of } E_j \text{ receives colour } \alpha$$
$$= 0 \text{ otherwise.}$$

The vector $\mathbf{y}(\alpha) = (y_1(\alpha),y_2(\alpha), \ldots ,y_m(\alpha))$ has coordinates 0,1, and is contained in Q. Further

$$\mathbf{z} = \frac{1}{k}(p_1,p_2,\ldots,p_m) = \frac{1}{k}\sum_{\alpha=1}^{k}\mathbf{y}(\alpha).$$

As the vector $\mathbf{z}$ is an extreme point of the polyhedron Q and as $\mathbf{y}(\alpha) \in Q$ we deduce: $y(1) = y(2) = \cdots = y(k)$. Then $\mathbf{z} = \mathbf{y}(1)$ and consequently $\mathbf{z}$ is a 0,1 vector, so (2) holds.

(2) implies (3). Clear.

(3) implies (4). For $\mathbf{d} \in \mathbb{N}^m$ consider the set

$$\overline{Q} = \{\mathbf{z}/\mathbf{z} \in Q, \mathbf{z} \in \mathbb{N}^m, \langle\mathbf{d},\mathbf{z}\rangle = \max_{y \in Q}\langle\mathbf{d},\mathbf{y}\rangle\}.$$

As $\overline{Q} \neq \emptyset$ from (3), and as $\overline{Q}$ is contained in a facet of the polyhedron Q, there exists a row vector $\mathbf{a}^{i_1}$ of the matrix such that

$$\langle\mathbf{a}^{i_1},\mathbf{z}\rangle = 1 \qquad (\mathbf{z} \in \overline{Q})$$

In other words, in H, every maximum $\mathbf{d}$-value matching covers the vertex x_{i_1}. Set

$$d_j^1 = d_j - 1 \text{ if } E_j \text{ appears in an optimal matching and contains } x_{i_1}$$
$$= d_j \text{ otherwise.}$$

It follows that $\mathbf{d}^1 = (d_1^1, d_2^1, \ldots, d_m^1) \geq \mathbf{0}$ and that $N\text{--}max\langle\mathbf{d}^1,\mathbf{y}\rangle = N\text{--}max\langle\mathbf{d},\mathbf{y}\rangle$. As before, there exists a vector $\mathbf{d}^2 \geq \mathbf{0}$ satisfying $N\text{--}max\langle\mathbf{d}^2,\mathbf{y}\rangle = N\text{--}max\langle\mathbf{d}^1,\mathbf{y}\rangle - 1$.

Continuing in this way, we arrive at $\mathbf{d}^k$ such that

$$N\text{--}max\langle\mathbf{d}^k,\mathbf{y}\rangle = 0.$$

Thus we have determined a sequence $(x_{i_1},x_{i_2},\ldots,x_{i_k})$ which contains, say, the vertex x_1 exactly t_1 times, x_2 exactly t_2 times, etc.

Observe that the vector $\mathbf{t} = (t_1,t_2,\ldots,t_n)$ is a $\mathbf{d}$-transversal of H; further

$$\sum_{i=1}^{n} t_i = k = \underset{y \in Q}{N\text{--}max}\langle\mathbf{d},\mathbf{y}\rangle.$$

From the duality theorem in linear programming, $\mathbf{t}$ is a minimum value $\mathbf{d}$-transversal, whence

$$\underset{y \in Q}{N\text{--}max}\langle\mathbf{d},\mathbf{y}\rangle = \Sigma t_i = \underset{t \in P(d)}{N\text{--}max}\langle\mathbf{1},\mathbf{t}\rangle.$$

Thus (4) holds.

(4) implies (5). Let $H' \subset H$ be a partial hypergraph of H; set $d_j = 1$ if $E_j \in H'$ and $d_j = 0$ otherwise. The vector $\mathbf{d} = (d_1, d_2, \ldots, d_m)$ satisfies

$$\underset{y \in Q(1)}{N\text{-}max} \langle \mathbf{d}, \mathbf{y} \rangle = \nu(H')$$

$$\underset{t \in P(d)}{N\text{-}min} \langle \mathbf{1}, \mathbf{t} \rangle = \tau(H').$$

Thus (4) implies $\nu(H') = \tau(H')$, whence (5).

(5) implies (1). Let $H = (E_1, E_2, \ldots, E_m)$ be a hypergraph on X satisfying (5). Let $\overline{H} = (E_1, E_2, \ldots, \overline{E}_m)$ be a hypergraph on the set of matchings of H where $\overline{E}_j$ denotes the family of matchings of H which contain the edge E_j. Clearly, $\overline{E}_j \cap \overline{E}_k = \emptyset$ if and only if $E_j \cap E_k \neq \emptyset$.

As H has the Helly property by virtue of (5), we have

$$\nu(\overline{H}) = \Delta(H)$$

$$q(\overline{H}) = \tau(H)$$

Further

$$\tau(\overline{H}) = q(H)$$

$$\Delta(\overline{H}) = \nu(H).$$

As H satisfies the König property, we deduce that $q(\overline{H}) = \Delta(\overline{H})$; for the same reason, every partial hypergraph of $\overline{H}$ has the coloured edge property. As we have already shown that (1) implies (5) we see that $\nu(\overline{H}) = \tau(\overline{H})$, i.e. $q(H) = \Delta(H)$, and (1) follows.

Corollary 1. *A hypergraph H is normal if and only if H satisfies the Helly property and $L(H)$ is a perfect graph.*

Indeed, if H is normal, it has the Helly property, since from (5) an intersecting family H' satisfies $\tau(H') = \nu(H') = 1$. Further, as $q(H) = \Delta(H)$, we have $\gamma(H^*) = r(H^*)$, and the 2-section $G = [H^*]_2$ satisfies $\gamma(G) = \omega(G)$. This equality being satisfied (for the same reason) for every induced subgraph of G, the graph G is "perfect" (cf. *Graphs*, §3, Chap. 16).

Conversely, if H has the Helly property and if G is its representative graph, the maximal edges of H^* are the maximal cliques of G (Proposition 1, §8, Chap. 1). If G is perfect, then $\gamma(G) = \omega(G)$, whence $\gamma(H^*) = r(H^*)$, whence $q(H) = \Delta(H)$.

This equality being satisfied for the same reason for every $H' \subset H$, the hypergraph H is normal.

Q.E.D.

It should be noted that H need not be normal if we do not assume the Helly property (cf. for example the hypergraph H_2 of Figure 8, Chap. 1).

Corollary 2. *Every co-arboreal hypergraph is normal.*

Indeed, let H be a co-arboreal hypergraph; it satisfies the Helly property, and $L(H)$ is a triangulated graph from the corollary to Theorem 12. Since every triangulated graph is perfect (cf. *Graphs* §, Chap. 6), Corollary 1 shows that H is normal.

6. Mengerian Hypergraphs

A hypergraph H is said to be *Mengerian* if for every $\mathbf{c} \in \mathbb{N}^n$ we have

$$(1) \qquad \underset{y \in Q(c)}{N\text{-}max} \langle \mathbf{1},\mathbf{y} \rangle = \underset{t \in P(1)}{N\text{-}min} \langle \mathbf{c},\mathbf{t} \rangle .$$

Observe that every balanced hypergraph is Mengerian (by Theorem 10). The converse is not true: for example a Mengerian hypergraph may have a chromatic number greater than 2, as does the hypergraph of Figure 17.

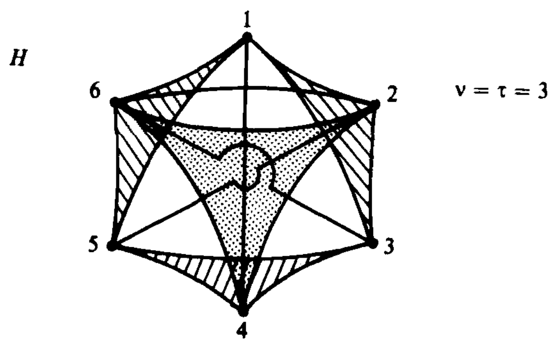


Figure 17. *A non-bicolorable Mengerian hypergraph.*

Proposition 1. *Let H be a Mengerian hypergraph, and let A be a set of vertices containing at least one edge; then the partial hypergraph $H/A = (E_i/E_i \subset A)$ is Mengerian.*

Proof. Let $c_i \geq 0$ be an integer defined for every vertex x_i of H/A. Set $\bar{c}_i = c_i$ if x_i is a vertex of H/A and $\bar{c}_i = 0$ otherwise; if $\overline{P}$ and $\overline{Q}$ denote the polyhedra associated with the hypergraph H/A, we have:

$$(1) \qquad N\!-\!\max_{y \in \overline{Q}(c)} \langle \mathbf{1},\mathbf{y} \rangle = N\!-\!\max_{y \in Q(\bar{c})} \langle \mathbf{1},\mathbf{y} \rangle$$

$$(2) \qquad N\!-\!\min_{t \in \overline{P}(1)} \langle \mathbf{c},\mathbf{t} \rangle = N\!-\!\min_{t \in P(1)} \langle \bar{\mathbf{c}},\mathbf{t} \rangle.$$

As H is Mengerian, the numbers (1) and (2) are equal, so the hypergraph H/A is Mengerian.

Q.E.D.

Proposition 2. *Let H be a Mengerian hypergraph, and let A be a set of vertices meeting all of the edges; then the induced subhypergraph $H_A = (E_i \cap A / i \leq m, E_i \cap A \neq \emptyset)$ is Mengerian.*

Proof. Let $c_i \geq 0$ be defined for every vertex $x_i \in A$. Set $\bar{c}_i = c_i$ if $x_i \in A$ and $\bar{c}_i = +\infty$ otherwise. If $\overline{P}$ and $\overline{Q}$ denote the polyhedra associated with the hypergraph H_A we have

$$(1) \qquad N\!-\!\max_{y \in \overline{Q}(c)} \langle \mathbf{1},\mathbf{y} \rangle = M\!-\!\max_{y \in Q(\bar{c})} \langle \mathbf{1},\mathbf{y} \rangle$$

$$(2) \qquad N\!-\!\min_{t \in P(1)} \langle \mathbf{c},\mathbf{t} \rangle = N\!-\!\min_{t \in P(1)} \langle \bar{\mathbf{c}},\mathbf{t} \rangle$$

As H is Mengerian the numbers (1) and (2) are equal. The hypergraph H_A is thus Mengerian.

Let H be a hypergraph, and let $\lambda \geq 0$ be an integer. We shall say that we *expand the vertex x by λ* if we replace x by λ new vertices $x^1, x^2, \ldots, x^\lambda$, and replace each edge E which contains x by λ new edges $E^1 = (E - \{x\}) \cup \{x^1\}$, $E^2 = (E - \{x\}) \cup \{x^2\} \cdots$. Expanding the vertex x by $\lambda = 0$ will be taken to mean deleting the vertex x and all the edges of H containing x.

Let $\mathbf{c} = (c_1,c_2,...,c_n)$ be a vector each of whose coordinates c_i is an integer ≥ 0. The *expansion* of H *by* $\mathbf{c}$ is the hypergraph H^c obtained from H by successively expanding vertex x_1 by c_1, x_2 by c_2, etc.

Theorem 17. *Let H be a hypergraph with m edges and n vertices. Let $\mathbf{c} = (c_1,c_2,...,c_n) \in \mathbb{N}^n$, and let $k \geq 1$ be an integer. Then*

$$(1) \qquad \nu_k(H^c) = \max\{<\mathbf{1},\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq k\mathbf{c}\},$$

$$(2) \qquad \tau_k(H^c) = \min\{<\mathbf{c},\mathbf{t}>/\mathbf{t} \in \mathbb{N}^n, A^*\mathbf{t} \geq k\mathbf{1}\}$$

$$(3) \qquad \tau^*(H^c) = \max_{y \in Q(c)} <\mathbf{1},\mathbf{y}> = \min_{t \in P(1)} <\mathbf{c},\mathbf{t}>.$$

Proof. It suffices to show (1) and (2) for the hypergraph H^c obtained by expanding the vertex x_1 by $\lambda = 0$ (suppression) or by $\lambda = 2$ (doubling), i.e. for $\mathbf{c} = (0,1,1,...,1)$ or for $\mathbf{c} = (2,1,1,...,1)$.

Proof of (1) with $\mathbf{c} = (0,1,1,...,1)$. Consider a $k\mathbf{c}$-matching $\bar{\mathbf{y}} = (\bar{y}_1,\bar{y}_2, \ldots ,\bar{y}_m)$ of H of maximum value $\Sigma\bar{y}_i$. Since $\bar{y}_j = 0$ for each edge E_j containing x_1, the vector $\bar{\mathbf{y}}$ determines a k-matching of H^c of value $\Sigma\bar{y}_j$, whence

$$\nu_k(H^c) \geq \sum_{j \geq 1} y_j = \max\{<\mathbf{1},\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq k\mathbf{c}\}.$$

Further, in H^c a k-matching $y = (y_1,y_2,...,y_m)$ of maximum value determines in H a $k\mathbf{c}$-matching of value Σy_j whence

$$\max\{<\mathbf{1},\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq k\mathbf{c}\} \geq \Sigma y_j = \nu_k(H^c).$$

Combining these inequalities we obtain (1).

Proof of (1) with $\mathbf{c} = (2,1,1,...,1)$. Let $(\bar{y}_1,\bar{y}_2, \ldots ,\bar{y}_m)$ be a maximum value $k\mathbf{c}$-matching. In H^c there are two vertices x_1' and x_1'' corresponding to a single vertex x_1 of H, and the set of edges $\{E_j/j \in J\}$ of H containing x_1 corresponds in H^c to two sets $\{E_j'/j \in J\}$ and $\{E_j''/j \in J\}$; we have

$$\sum_{j \in J} \bar{y}_j \leq 2k.$$

Consider a vector

$$\mathbf{y} = (y_j / j \in \{1,2,\ldots,m\} - J)\cdot(y'_j / j \in J)\cdot(y''_j / j \in J)$$

where

$$y'_j + y''_j = \bar{y}_j \qquad (j \in J)$$

$$\sum_{j \in J} y'_j \leq k$$

$$\sum_{j \in J} y''_j \leq k.$$

This vector being a k-matching of H^c we have

$$\nu_k(H^c) \geq \Sigma y_i = \sum_{j=1}^{m} \bar{y}_j = \max\{<\mathbf{1},\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq k\mathbf{c}\}.$$

Thus (1) follows.

Proof of (2) with $\mathbf{c} = (0,1,1,\ldots,1)$. Consider a k-transversal $(t_1,t_2,\ldots,t_n)$ of H of minimum c-value $\sum_{i \neq 1} t_i$. As the vector $(t_2,t_3,\ldots,t_n)$ is a k-transversal of H^c we have

$$\tau_k(H^c) \leq \sum_{i \neq 1} t_i = \min\{<\mathbf{c},\mathbf{t}>/\mathbf{t} \in \mathbb{N}^n, A^*\mathbf{t} \geq k\mathbf{1}\}$$

Conversely, if $(t_2,t_3,\ldots,t_n)$ is a minimum k-transversal of H^c, the vector $(k,t_2,t_3,\ldots,t_n)$ is a k-transversal of H, whence

$$\min\{<\mathbf{c},\mathbf{t}>/\mathbf{t} \in \mathbb{N}^n, A^*\mathbf{t} \geq k\mathbf{1}\} \leq \sum_{i \neq 1} t_i = \tau_k(H^c).$$

By combining these inequalities we obtain (2).

Proof of (2) with $c = (2,1,1,\ldots,1)$. Consider an optimal k-transversal $(t_1,t_2,\ldots,t_n)$ of H with minimum c-value $2t_1+t_2+\cdots+t_n$. Since the vector $(t_1,t_1,t_2,t_3,\ldots,t_n)$ is a k-transversal of H^c we have

$$\tau_k(H^c) \leq 2t_1+t_2+\cdots+t_n = \min\{<\mathbf{c},\mathbf{t}>/\mathbf{t} \in \mathbb{N}^n, A^*\mathbf{t} \geq k\mathbf{1}\}.$$

Conversely, if $(t_1,t'_1,t_2,\ldots,t_n)$ is a minimum k-transversal of H^c, we have $t'_1 = t_1$. Since the vector $(t_1,t_2,\ldots,t_n)$ is a k-transversal of H, we have

$$\min\{<\mathbf{c},\mathbf{t}>/\mathbf{t}\in\mathbb{N}^n, A^*\mathbf{t}\geq k\mathbf{1}\}\leq 2t_1+t_2+\cdots+t_n = \tau_k(H^c).$$

Combining these inequalities we obtain (2).

Proof of (3). Let k tend to infinity in $\frac{1}{k}\tau_k(H^c)$ or $\frac{1}{k}\nu_k(H^c)$. We obtain $\tau^*(H^c)$. Hence (1) and (2) imply (3).

Corollary. *Let $\mathcal{H}$ be a family of hypergraphs with the König property, which further satisfy:*

$$H\in\mathcal{H},\ \mathbf{c}\in\mathbb{N}^{n(H)}\Rightarrow H^c\in\mathcal{H}.$$

Then every hypergraph of $\mathcal{H}$ is Mengerian.

Proof. Clear.

Example 1 (Menger). Let G be a multigraph and let a,b be two vertices of G. Denote by H the hypergraph on the set of edges of G, having as edges the simple paths joining a and b. A transversal of H is then a simple cocycle $\omega(S)$ of G with $a\in S$ and $b\in X-S$.

Menger's theorem implies that H has the König property. Further, expanding a vertex of H by $\lambda\geq 0$ becomes replacing an edge of G by λ parallel edges: thus H is a Mengerian hypergraph.

Example 2 (Menger). Let G be a simple graph and let a,b be two non-adjacent vertices. Denote by H the hypergraph on the set of vertices of G different from a,b, and having as edges the sets of intermediate vertices of simple paths joining a and b. A minimal tranversal of H is then a minimal cut-set disconnecting a and b, and Menger's second theorem shows that H has the König property. Furthermore, expanding a vertex of x by $\lambda\geq 0$ corresponds to replacing the vertex x in G by an independent set of λ elements, each joined to all the neighbours of x. Thus H is a Mengerian hypergraph.

Example 3 (Edmonds). Let G be a multigraph on X, and let S be a subset of X having at least two elements. Denote by H the hypergraph on the set of edges of G having as edges the simple paths of the form $\mu=[s_1,a_1,a_2,\ldots,a_k,s_2]$ with $s_1,s_2\in S$ and $a_1,a_2,\ldots,a_k\in X-S$. A theorem of Edmonds [1970] shows that H has the König property. Since expanding a vertex of H corresponds to multiplying an edge of G, H is a Mengerian hypergraph.

Example 4 (Edmonds [1973]). Let $G = (X,U)$ be a directed graph, and let a be a vertex of G which is an ancestor of all the others (a "root" of G). Denote by H the hypergraph on the set of arcs of G having as edges those arborescences rooted at a which cover all the vertices of G.

The transversals of H are the sets of arcs of the form $\omega^+(S)$ with $a \in S$, $S \neq X$ (i.e. which go from S to $X - S$). A theorem of Edmonds [1973] implies that H has the König property. Expanding a vertex of H by $\lambda = 0$ corresponds to eliminating an arc of G, and expanding by $\lambda > 0$ corresponds to replacing it by λ parallel arcs. Thus H is a Mengerian hypergraph.

For the extension of this example by replacing rooted arborescences by forests of arborescences, cf. Frank [1979].

Example 5. Let $G = (X,U)$ be a directed graph; denote by H the hypergraph on the set of arcs of G having as edges the cocircuits of G. A theorem of Lucchesi and Younger [1978] shows that H has the König property. Expanding a vertex of H by $\lambda = 0$ corresponds to contracting an arc of G, and expanding by $\lambda > 0$ corresponds to replacing an arc by a path of length λ. Thus H is a Mengerian hypergraph.

For further examples, cf. Woodall [1978], Seymour [1977], Maurras [1976]. A method of proving that these hypergraphs have the König property is, nonetheless, necessary; general ideas for such a method have been given by Lovász [1976] and extended by Schrijver and Seymour [1979].

Lemma 1 (Hoffman [1974]). *Let $A = ((a_j^i))$ be a matrix with n rows and m columns, with $a_j^i \in \mathbb{N}$. Let k be an integer ≥ 1. If the convex polyhedron $P = \{\mathbf{x}/\mathbf{x} \in \mathbb{R}^n, A^*\mathbf{x} \geq \mathbf{1}\}$, is such that the number $k \min_{x \in P} \langle \mathbf{c},\mathbf{x} \rangle$ is an integer for every $\mathbf{c} \in \mathbb{N}^n$, then the extreme points of P have coordinates multiples of $\frac{1}{k}$.*

Proof. Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be an extreme point of P; we shall show, for example, that y_1 is a multiple of $\frac{1}{k}$. Set $\mathbf{e}_1 = (1,0,0,\ldots,0)$. We show that there exists a vector $\mathbf{c} \in \mathbb{N}^n$ such that

(1) $$<\mathbf{c},\mathbf{y}> = \min_{x \in P} <\mathbf{c},\mathbf{x}>,$$

(2) $$<\mathbf{c}+\mathbf{e}_1,\mathbf{y}> = \min_{x \in P} <\mathbf{c}+\mathbf{e}_1,\mathbf{x}>.$$

Then, the hypothesis will imply that $y_1 = <\mathbf{c}+\mathbf{e}_1,\mathbf{y}> - <\mathbf{c},\mathbf{y}>$ is a multiple of $\frac{1}{k}$, which achieves the proof.

Set $I = \{i / y_i = 0\}$ and $J = \{j / <\mathbf{a}_j,\mathbf{y}> = 1\}$ and consider the vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, where

$$d_i = \begin{cases} \sum_{j \in J} a_j^i + 1 & \text{if } i \in I \\ \sum_{j \in J} a_j^i & \text{if } i \notin I \end{cases}$$

For every $\mathbf{x} \in P$ we have

$$<\mathbf{d},\mathbf{x}> = \sum_i d_i x_i = \sum_{i \in I} x_i + \sum_{j \in J} <\mathbf{a}_j,\mathbf{x}> \geq |J|$$

As equality holds for $\mathbf{x} = \mathbf{y}$,

$$<\mathbf{d},\mathbf{y}> = \min_{x \in P} <\mathbf{d},\mathbf{x}>$$

Further, as the hyperplanes $\{\mathbf{x}/x_i = 0\}$ for $i \in I$ and the hyperplanes $\{\mathbf{x}/<\mathbf{a}_j,\mathbf{x}>=1\}$ with $j \in J$ completely define the extreme point $\mathbf{y}$, we have also

(3) $$<\mathbf{d},\mathbf{x}> \; > \; <\mathbf{d},\mathbf{y}> \qquad (\mathbf{x} \neq \mathbf{y},\ \mathbf{x} \in P)$$

Suppose that for each integer $\lambda \geq 1$, the minimum of $<\lambda\mathbf{d}+\mathbf{e}_1,\mathbf{x}>$ for $\mathbf{x} \in P$ is attained at an extreme point $\mathbf{z}(\lambda) \neq \mathbf{y}$; as P has only a finite number of extreme points, there is an extreme point $\bar{\mathbf{x}}$ such that $\bar{\mathbf{x}} = \mathbf{z}(\lambda)$ for infinitely many values of λ, that is to say for infinitely many λ we have

$$<\mathbf{d},\bar{\mathbf{x}}> + \frac{1}{\lambda}\bar{x}_1 \leq <\mathbf{d},\mathbf{y}> + \frac{1}{\lambda} y_1$$

Thus $\bar{\mathbf{x}} \neq \mathbf{y}$, $\bar{\mathbf{x}} \in P$, and $<\mathbf{d},\bar{\mathbf{x}}> \leq <\mathbf{d},\mathbf{y}>$, contradicting (3). Hence for some $\lambda \geq 1$ the minimum of $<\lambda\mathbf{d}+\mathbf{e}_1,\mathbf{x}>$ is attained at $\mathbf{y}$. Since the minimum of $<\lambda\mathbf{d},\mathbf{x}>$ is also attained at $\mathbf{y}$, the vector $\mathbf{c} = \lambda\mathbf{d}$ must satisfy conditions (1) and (2), which completes the proof.

Lemma 2. *Let H be a hypergraph of order n, and let k be an integer ≥ 1; the following conditions are equivalent:*

(1) $k\tau^*(H^c)$ is an integer for every $\mathbf{c} \in \mathbb{N}^n$,

(2) $\tau^*(H^c) = \frac{1}{k}\tau_k(H^c)$ for every $\mathbf{c} \in \mathbb{N}^n$.

Proof. It suffices to show that (1) implies (2).

Let A be the incidence matrix of H. The polyhedron $P = \{\mathbf{x}/\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \geq \mathbf{0}, A^*\mathbf{x} \geq \mathbf{1}\}$ satisfies the conditions of lemma 1, so each of its extreme points has all coordinates a multiple of $\frac{1}{k}$. In particular, the minimum of $<\mathbf{c},\mathbf{x}>$ is attained at a point of the form $\mathbf{x}_0 = \frac{\mathbf{t}_0}{k}$ where $\mathbf{t}_0 = (t_1, t_2, \ldots, t_n) \in \mathbb{N}^n$. As $A^*\mathbf{x}_0 \geq \mathbf{1}$ the vector $\mathbf{t}_0$ is a k-transversal of H and further it has minimum $\mathbf{c}$-value. Thus

$$\frac{1}{k}\{\min <\mathbf{c},\mathbf{t}>/\mathbf{t} \in \mathbb{N}^n, A^*\mathbf{t} \geq k\mathbf{1}\} = \frac{1}{k}<\mathbf{c},\mathbf{t}_0>$$

$$= \min_{x \in P} <\mathbf{c},\mathbf{x}>.$$

From Theorem 17, this implies (2).

Lemma 3. *Let H be a hypergraph of order n, and let k be an integer ≥ 1. The following conditions are equivalent:*

(3) $\frac{1}{k}\,\nu_k(H^c) = \tau^*(H^c)$ for every $\mathbf{c} \in \mathbb{N}^n$;

(4) $\nu_k(H^c) = \tau_k(H^c)$ for every $\mathbf{c} \in \mathbb{N}^n$.

It suffices to show that (3) implies (4).

Indeed, (3) implies condition (1) of Lemma 2, and thus (2); further, (2) and (3) imply (4).

Theorem 18 (Lovász [1976]). *A hypergraph H of order n is Mengerian if and only if for an integer $q \geq 0$ we have*

$$(5) \qquad \frac{1}{q}\,\nu_q(H^c) = \nu(H^c) \quad (\mathbf{c} \in \mathbb{N}^n).$$

Proof. Suppose that for each $\mathbf{c} \in \mathbb{N}^n$ we have (5), that is

$$\min\{<1,\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} = q\mathbf{c}\} + q\min\{<1,\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq \mathbf{c}\}$$

Let $\mathbf{c} = q\mathbf{c}'$; we may write

$$\min\{<1,\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq q^2\mathbf{c}'\} = q\min\{<1,\mathbf{y}>/\mathbf{y} \in \mathbb{N}^m, A\mathbf{y} \leq q\mathbf{c}'\}$$

Hence, for every $\mathbf{c} \in \mathbb{N}^n$

$$\frac{1}{q^2}\,\nu_{q^2}(H^c) = \frac{1}{q}\,\nu_q(H^c) = \nu(H^c)$$

From Theorem 1, Chapter 3,

$$\tau^*(H^c) = \lim_{s \to \infty} \frac{1}{q^s}\nu_{q^s}(H^c) = \nu(H^c)$$

From Lemma 3 with $k = 1$, we obtain $\nu(H^c) = \tau(H^c)$. Thus H is Mengerian.

Q.E.D.

Let H be a hypergraph of order n. We say that the vertex x_1 is *multiplied* by an integer $\lambda \geq 0$ if we replace x_1 by a set $X_1 = \{x_1^1, x_1^2, \ldots, x_1^\lambda\}$ of λ new vertices and if we replace each edge E containing x_1 by an edge $\overline{E} = (E - \{x_1\}) \cup X_1$; multiplication of x_1 by $\lambda = 0$ becomes replacement of H by the subhypergraph of H induced by $X - \{x_1\}$.

Let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n$. The multiplication of H by $\mathbf{c}$ is the hypergraph $\overline{H}^{(c)}$ obtained by multiplying the vertex x_1 by c_1, x_2 by c_2, etc.

Remark. *If H is balanced then $\overline{H}^{(c)}$ is also balanced.* Indeed, for $\mathbf{c} = (0,1,1,\ldots,1)$ the hypergraph $\overline{H}^{(c)}$, which is a subhypergraph of H, is necessarily balanced (Proposition 1, §3). For $\mathbf{c} = (2,1,1,\ldots,1)$ the hypergraph $\overline{H}^{(c)}$ is obtained by replacing the vertex x_1 by $\{x_1', x_1''\}$; if $\overline{H}^{(c)}$ cannot contains an odd cycle $(a_1\overline{E}_1, a_2\overline{E}_1, \ldots, a_1)$ such that no $\overline{E}_i$ contains three a_i, then x_1' and x_1'' are both vertices of the cycle, and the two edges next to x_1' in the cycle also contain x_1'', so at least one contains three vertices of the sequence: a contradiction.

In contrast, *if H is balanced, its expansion H^c need not necessarily be balanced.* For example, consider the balanced hypergraph shown in Figure 12, and split the vertex f_1 into two vertices f_1' and f_1'': then no edge of the resulting hypergraph contains three vertices for the following odd cycle:

$$f_1'',\ \{f_1'',\ f_4\},\ f_4,\ \{f_1',f_2,f_3,f_4\},\ f_2,\ \{f_1'',f_2\},\ f_1''.$$

We shall study some conditions for the transversal hypergraph to be Mengerian. Let $\sigma(H)$ be the maximum number of colours for a colouring of the vertices of H such that each edge contains all the colours. Clearly

$$\sigma(H) \leq \min_j |E_j| = s(H).$$

We shall say that H has the *Gupta property* if $\sigma(\overline{H}^{(c)}) = s(\overline{H}^{(c)})$ for all $\mathbf{c} \in \mathbb{N}^n$. For example, the dual of a bipartite graph has the Gupta property, by Gupta's Theorem [1978].

Lemma. *Let H be a simple hypergraph of order n and let $K = Tr\,H$ be its transversal hypergraph. Then K is Mengerian if and only if H has the Gupta property.*

Proof. We see immediately that if $\mathbf{c} \in \mathbb{N}^n$ we have

(1) $$Tr\,\overline{H}^{(c)} = (Tr\,H)^c.$$

Further, for every hypergraph H,

(2) $$\sigma(H) = \nu(Tr\,H)$$

(3) $$s(H) = \tau(Tr\,H).$$

From (1), (2) and (3) we obtain

$$\sigma(\overline{H}^{(c)}) = \nu(Tr\,\overline{H}^{(c)}) = \nu[(Tr\,H)^c]$$
$$s(\overline{H}^{(c)}) = \tau(Tr\,\overline{H}^{(c)}) = \tau[(Tr\,H)^c]$$

H has the Gupta property if and only if these two quantities are equal, i.e. if H is Mengerian.

Theorem 19 (Berge [1984]). *Let H be a simple balanced hypergraph; then $Tr\,H$ is a*

Mengerian hypergraph.

Indeed, if H is balanced, the hypergraph $\overline{H}^{(c)}$ is also balanced; thus, from Corollary 2 to Theorem 8, we have $\sigma(\overline{H}^{(c)}) = s(\overline{H}^{(c)})$. From the lemma, this implies that $Tr\,H$ is Mengerian.

Remark. The converse of Theorem 19 is not true; for example, if H is the dual hypergraph of K_4 (Figure 19), then $Tr\,H$ is the Mengerian hypergraph of Figure 17, but it is clear that H is not balanced. Nonetheless, if $Tr\,H'$ is Mengerian for all $H' \subset H$, then every $H' \subset H$ has the Gupta property and H is necessarily balanced.

7. Paranormal Hypergraphs

We may generalise Mengerian hypergraphs. Observe first of all the equivalence of the following properties:

(1) every extreme point of the polyhedron $P = \{\mathbf{t}/\mathbf{t} \in \mathbb{R}^n, \mathbf{t} \geq \mathbf{0}, A^*\mathbf{t} \geq \mathbf{1}\}$ is a vector with integer coordinates;

(2) $\min\limits_{t \in P(1)} \langle \mathbf{c},\mathbf{t} \rangle$ is an integer for every $\mathbf{c} \in \mathbb{N}^n$;

(3) $N\text{-}\min\limits_{t \in P(1)} \langle \mathbf{c},\mathbf{t} \rangle = \min\limits_{t \in P(1)} \langle \mathbf{c},\mathbf{t} \rangle$ for every $\mathbf{c} \in \mathbb{N}^n$.

The equivalence of (1) and (2) follows from Lemma 1 (with $k = 1$); the equivalence of (2) and (3) follows from Lemma 2 (with $k = 1$). We shall say that a hypergraph H is *paranormal* if it satisfies (1) or, equivalently, (2) or (3). (These hypergraphs, which were first studied by Fulkerson, are also called "fulkersonian" by Schrijver, or "having the weak max-flow min-cut property" by Seymour).

If a hypergraph H is Mengerian then it satisfies (3) and is thus paranormal; the converse is not true, as may be seen in Figure 19: the hypergraph $(K_4)^*$, dual of K_4, is paranormal but not Mengerian (since $\tau \neq \nu$).

Seymour [1977] conjectured that if a simple paranormal hypergraph cannot be reduced to $(K_4)^*$ by means of the operations H/A and H_A described in Propositions 1 and 2, §6, then H is Mengerian.

We shall now give some examples of paranormal hypergraphs.

Example 1 (Seymour [1977]). Let G be a planar graph; let $H(G)$ be the hypergraph whose vertices are the edges of G and whose edges are the elementary odd cycles of G. Then Seymour has shown that $H(G)$ is paranormal. In contrast, for $G = K_5$, which is

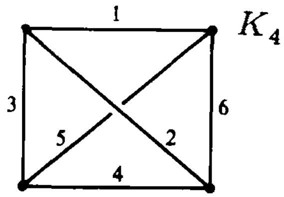
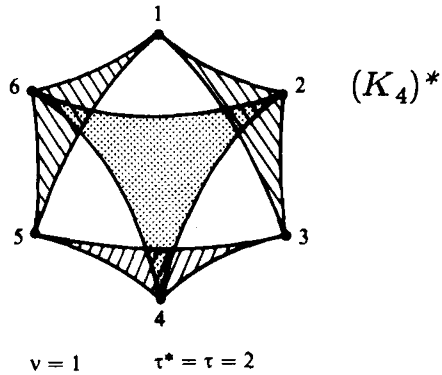


Figure 18



A paranormal non-Mengerian hypergraph.

Figure 19

non-planar, $H(K_5)$ is not paranormal.

Example 2 (Hu [1963]). Let G be a graph, s,s',t,t' four vertices of G, and let $H(G)$ be the hypergraph whose vertices are the edges of G and whose edges are the simple paths joining s and s', or joining t and t'. Hu has shown that $H(G)$ is paranormal (a result known as the "two-commodity flow theorem"). The hypergraph $H(G)$ need not be Mengerian as may be seen from the graph G of Figure 20 for which $H(G)$ is none other than the non-Mengerian hypergraph of Figure 19.

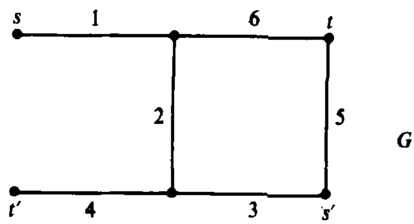


Figure 20

Theorem 20 (Lehman, Fulkerson). *Let H be a simple hypergraph and let $K = Tr\, H$ be its transversal hypergraph. Then H is paranormal if and only if we have*

$$(1) \qquad \tau(H^w)\tau(K^c) \leq <\mathbf{c},\mathbf{w}> \quad (\mathbf{c},\mathbf{w} \in \mathbb{N}^n).$$

Recall that (1) is sometimes known as the *width-length* inequality for the following reason: let G be a network flow with a source a and a sink z; if c_i denotes the "length" of edge i and w_i its "width", the hypergraph H, whose vertices are the edges of G and whose edges are the paths between a and z, gives us the following interpretation:

$\tau(K^c) = \min_{E \in H} \sum_{i \in E} c_i$ is the length of a shortest path from a to z,

$\tau(H^w) = \min_{T \in Tr\, H} \sum_{i \in T} w_i$ is the width of a smallest cut between a and z.

The proof of (1) given by Lehman [1975] is valid for all paranormal hypergraphs and Fulkerson extended it to matrices with non-integer entries using the theory of pairs of "blocking" matrices(1).

Corollary. *Let H be a simple paranormal hypergraph; then $Tr\, H$ is also a paranormal hypergraph.*

Indeed, for $K = Tr\, H$ inequality (1) may be rewritten as

$$\tau(K^w)\tau([Tr\, K]^c) \leq <\mathbf{w},\mathbf{c}> \quad (\mathbf{w},\mathbf{c} \in \mathbb{N}^n)$$

Thus K is paranormal.

Remark. If H is Mengerian, the preceding corollary shows that $Tr\, H$ is paranormal; nonetheless $Tr\, H$ need not be Mengerian: for example the Mengerian hypergraph of Figure 17 has as its transversal hypergraph that of Figure 19, which is not Mengerian. If H is balanced we also know that $Tr\, H$ is paranormal (from Theorem 18). In the case when H is normal, the hypergraph $Tr\, H$ need not be normal: for example, the normal

(1) For the matrix proof, cf. Fulkerson [1981]. If A is a matrix $a_j^i \geq 0$, a_j^i real (and not necessarily integer) and such that no column vector is a convex linear combination of the others, its "blocking" matrix B is a matrix whose column vectors are the extreme points of the polyhedron

$$P = \{t \mid t \in \mathbb{R}^n, t \geq 0, A^* t \geq 1\}$$

It is easily seen that the matrices A and B play a symmetric role. When A is the incidence matrix of a hypergraph H, the matrix B is the incidence matrix of $Tr\, H$ if and only if H is paranormal.

hypergraph H of Figure 16 satisfies $\tau^*(Tr\,H) = \frac{3}{2}$, so $Tr\,H$ is not paranormal.

An important family of paranormal hypergraphs appears in Graph Theory: these are the "S-joints" (introduced by Little [1973] to generalise an idea of Kasteleyn), and the "S-cuts" (considered by Lovász in 1977).

Let $G = (X,E)$ be a multigraph which we suppose for simplicity to have no loops and to be connected; let $S \subset X$ be a non-empty set of vertices.

We call an *S-joint* of G a set of edges $F \subset E$ forming a partial graph $G' = (X,F)$ whose set of vertices of odd degree coincide with S, with F being minimal for this property.

Observe that an S-joint of G exists if and only if $|S|$ is even. Indeed, if $|S|$ is even, divide S into disjoint pairs $\{s_1,s_1'\}$, $\{s_2,s_2'\}$, etc., and consider for each i a chain μ_i joining s_i and s_i'. The edges of G which belongs to an odd number of μ_i form an S-joint.

Conversely, if there exists an S-joint F, then the partial graph $G' = (X,F)$ satisfies, modulo 2,

$$|S| \equiv \sum_{x \in S} d_{G'}(x) \equiv \sum_{x \in X} d_{G'}(x) \equiv 2m(G') \equiv 0$$

We shall study the hypergraph of S-joints of G which we denote by H^s.

Recall some classical notation from Graph Theory. Let $G = (X,E)$ be a multigraph on X, and let $A \subset X$. The *cocycle* $\omega(A)$ is the set of edges of G joining A to its complement $X-A$; a cocycle is *elementary* if it contains no other cocycles, or equivalently if G_A and G_{X-A} are connected. Further, $\omega(A) = \omega(X-A)$. If $S \subset X$, an *S-cut* of G is an elementary cocycle $\omega(A)$ for which $|S \cap A|$ and $|S \cap (X-A)|$ are both odd.

Observe that an S-cut of G exists if and only if $|S|$ is even. We shall study the hypergraph of S-cuts of G which we denote K^s.

Example 1. Let $G = (X,E)$ be a connected multigraph of even order. An X-joint of G (being minimal) cannot contain a cycle of G; thus it is a forest of G. Further, each vertex of this forest has odd degree. In particular, a perfect matching of G, if it exists, is an X-joint of G.

On the other hand, an X-cut is nothing but an elementary cocycle $\omega(A)$ for which $|A|$ is odd.

Example 2. Let $G = (X,E)$ be a transporation network with source $a \in X$ and sink $z \in X$, and a capacity associated with each edge. Set $S = \{a,z\}$. An S-joint is a simple path between a and z, and an S-cut is a "cut" between a and z.

Example 3. Let G be a connected multigraph on X with a length associated with each edge, and let S be the set of vertices x with $d_G(x)$ odd. An S-joint is a minimal set of edges which must be doubled to obtain an eulerian multigraph.

An S-joint of minimum total length defines the edges to be traversed twice in the "chinese postman problem" (Guan Meigu) well known in Operations Research. An S-cut is an elementary cocyle $\omega(A)$ with $|A\cap S|$ odd, i.e. satisfying, modulo 2,

$$|\omega(A)| \equiv \sum_{x\in A\cap(X-S)} d_G(x) + \sum_{x\in A\cap S} d_G(x) \equiv |A\cap S| \equiv 1$$

Proposition. *Let G be a connected multigraph, and let S be a set of vertices with $|S|$ even. Then the hypergraph H^s of S-cuts is the transversal hypergraph of the hypergraph K^s of S-joints.*

Proof.

1. First we shall show that if $E \in H^s$ and $F \in K^s$, then $E \cap F \neq \emptyset$.

Indeed, otherwise we have $E \cap F = \emptyset$, $E \in H^s$, $F = \omega(A)$, where $|S\cap A|$ and $|S\cap(X-A)|$ are odd. Since E is the union of edge-disjoint paths μ_i between pairs $\{s_i,s_i'\}$ forming a partition of S, and since none of the μ_i meet $\omega(A)$, this implies that $|S\cap A|$ and $|S\cap(X-A)|$ are even: contradiction.

2. Let $F_0 \in Tr\,H^s$. Since F_0 meets all the $E \in H^s$, the partial graph $G-F_0$ does not allow S to be joined in pairs, and thus it has several connected components $X_1,X_2,\ldots,X_k$; further, at least one of the $|S\cap X_i|$ is odd (otherwise we may join the vertices in pairs). Since G is connected, we have $F_0 \supset \omega(S\cap X_i)$; thus F_0 contains an S-cut F. From the minimality of the transversal F_0, and from part 1, we have $F_0 = F$. Thus every minimal transversal of H^S is as S-cut, which achieves the proof.

Lovász-Seymour Theorem. *Let G be a connected multigraph, and let S be a set of vertices with $|S|$ even. Then H^s and K^s are paranormal hypergraphs.*

The fact that K^s is paranormal was shown by Lovász [1977], and that H^s is paranormal by Seymour [1977]. In fact these two theorems are clearly equivalent by virtue of Proposition 1. Further Lovász [1977] has shown that

$$\nu_{2k}(K^s) = k\nu_2(K^s).$$

Remark. H^s and K^s are not, in general, Mengerian. For example, if G is a cubic graph without bridges, of chromatic index 4 (such as Petersen's graph), Seymour has shown that H^X cannot have the König property and so certainly is not Mengerian. By contrast, H^X is Mengerian if it cannot be reduced to $(K_4)^*$ in the sense of Seymour's conjecture (Seymour [1977]).

Exercises on Chapter 5

Exercise 1 (§1)

If $r(H) > 3$, it is not true that every B-cycle contains a B-cycle such that every pair of non-consecutive edges are disjoint. Show this for the B-cycle of length 7 defined by the sequence of edges:

(12, 2390, 34, 45, 5690, 678, 781).

Exercise 2 (§1)

Sterboul [1973] has conjectured that if $\chi(H) > 2$ there exists a B-cycle such that every pair of non-consecutive edges is disjoint. Show that we cannot suppose that the B-cycle has the further property that two consecutive edges have exactly one vertex in common: for example take the hypergraph K^r_{2r-1}.

Exercise 3 (§1)

Show that example 2 is a special case of example 3 (§2), but that example 4 (§2) cannot be considered a special case of example 3 (which relies on a theorem of Tutte on graphic matroids).

Exercise 4 (§2)

Let P_n be a graph on X which consists of an elementary path of n vertices.

Let H_n be the hypergraph on X whose edges are the maximal cliques of the complement $\overline{P}_n$. Show that for $n \leq 6$, H_n is unimodular (Chvatal).

Hint: reduce to example 4 by an appropriate choice of a tree.

Exercise 5 (§2) Show that Ghouila-Houri's Theorem may be applied to extend Theorem 5 in the following way: if a matrix A of 0's, 1's, and -1's has no square submatrix of order $2k+1$ each of whose entries is greater than or equal to the corresponding entry of B_{2k+1} (the incidence matrix of the cycle C_{2k+1}), then A is totally unimodular.

(Another proof has been given by Commoner [1973], and Yannakakis [1980] has given an efficient algorithm to find a maximum matching in this case).

Exercise 6 (§2)

Let G be a bipartite graph. Let H be a hypergraph on the edge-set E of G whose edges are E and the complete stars of G. Show that H is unimodular.

Exercise 7 (§3)

Meyniel has conjectured that for every hypergraph H, the relation

$$\chi(H_A) \leq k \qquad (A \subset X)$$

implies $\tau(H) \leq (k-1)\nu(H)$. This is always true for $k = 2$, from Theorem 9; further if H is a partial hypergraph of the complete multipartie hypergraph, this reduces to the conjecture of Ryser.

Exercise 8 (§3)

Show that if A is a totally balanced incidence matrix, the matrix A^*A (boolean matrix product of A with its transpose A^*) is also a totally balanced matrix, as is the k-th boolean power A^k (Lubiw [1985]).

Exercise 9 (§3)

Show that if $H = (E_1, E_2, \ldots, E_m)$ is a totally balanced hypergraph on X, then $H+(X)$ and $H+(E_1 \cap E_2)$ are also totally balanced hypergraphs.

Exercise 10 (§3)

Using the preceding exercise, show that a totally balanced hypergraph of order n without repeated edges has at most $\binom{n}{2}+n$ edges; further every maximal totally balanced hypergraph without repeated edges has exactly $\binom{n}{2}+n$ edges (Anstee [1985]).

For a simpler proof, cf. Lehel [1985].

Exercise 11 (§4)

Let H be a hypergraph and let $L(H)$ be the representative graph of H with weight $|E_i \cap E_j|$ associated with each edge $[e_i, e_j]$. Let $F \subset L(H)$ be a maximum weight forest. Show that

$$\mu(H) = \sum_{i=1}^{n} [p(F_{X_i}) - 1]$$

where $X_i = \{e_j / x_i \in E_j \text{ in } H\}$ and where $p(F_{X_i})$ denotes the number of connected components of the subgraph of F induced by X_i (Lewin [1983]).

Exercise 12 (§7)

Lovász has shown: "If a digraph G has at most k pairwise disjoint co-circuits, then each family (with repetition) of co-circuits covering each arc at most twice is of cardinality $\leq 2k$". Show that this implies a generalisation of a theorem of Lucchesi and Younger: "If in a digraph G, we associate with each edge i an integer weight $c_i \geq 0$, then the minimum weight of a set of arcs which meet every cocircuit is equal to the maximum number of cocircuits forming a family using the arc i at most c_i times for $i = 1,2,...,m$".

Exercise 13 (§7)

As an analogue of Lemma 3, Theorem 17, Schrijver has conjectured that the following conditions are equivalent:

(i) $\quad \frac{1}{k}\tau_k(H') = \tau^*(H') \quad (H' \subset H)$

(ii) $\quad \nu_k(H') = \tau_k(H') \quad (H' \subset H)$

The equivalence of (i) and (ii), proved by Lovász [1977] for $k = 1,2,3$ is false for $k = 60$ (Schrijver, Seymour [1979]). Show this for the hypergraph H on $X = \{1,2,...,9\}$ whose edges are

$$E_1 = X - \{1,3,5\}$$

$E_2 = X-\{1,4,6\}$

$E_3 = X-\{2,3,6\}$

$E_4 = X-\{2,4,5\}$

$E_5 = X-\{7\}$

$E_6 = X-\{8\}$

$E_7 = X-\{9\}$

Show that $\tau_{60}(H') = 60\tau(H')$ and $\nu_{60}(H) \neq 60\tau^*(H)$.

Exercise 14 (§7)

Show that the following conditions are equivalent:

(i) $\tau^*(H^c)$ is an integer $(\mathbf{c} \in \{0,1\}^n)$

(ii) $\tau^*(H^c) = \tau(H^c)$ $(\mathbf{c} \in \{0,1\}^n)$

Hint: If (i) is true and (ii) is false, consider a hypergraph of minimum order such that (ii) is false.

Exercise 15 (§7)

Deduce from the preceding exercise that the following are equivalent:

(i) $\nu(H^c) = \tau(H^c)$ $(\mathbf{c} \in \{0,1\}^n)$

(ii) $\nu(H^c) = \tau^*(H^c)$ $(\mathbf{c} \in \{0,1\}^n)$.

Appendix

Matchings and Colourings in Matroids

The concept of a matroid, introduced by Whitney in 1935 in order to generalise linear independence allows us to restate a large number of theorems in optimization theory. First of all, it has been observed by many authors that the hypergraph of *independent sets* is such that one may use Kruskal's greedy algorithm to determine a tree of maximum weight. The identification of regular matroids with unimodular hypergraphs is due to Tutte, Camion, and to Seymour, who also showed that if $\mathcal{C}$ is the family of circuits of a matroid and if e is an element of the matroid then the hypergraph $\{C-e/C\in\mathcal{C}, e\in C\}$ is mengerian if and only if the matroid is linear and does not contain Fano's matroid as a minor[1].

We shall consider here the concepts of matching and colouring defined for hypergraphs in the preceding chapters.

Let $E = \{e_1,e_2,...,e_m\}$ be a finite set, and let $\mathcal{F}$ be a set of subsets of E. We shall say that $\mathcal{F}$ constitutes a *matroid on* E if

(1) $\{e_i\} \in \mathcal{F}\ (i = 1,2,...,m)$

(2) $F \in \mathcal{F},\ F' \neq \emptyset,\ F' \subset F \Rightarrow F' \in \mathcal{F}$

(3) For each $S \subset E$, if F and F' are two members of $\mathcal{F}$ contained in S and maximal with this property, then $|F| = |F'|$.

The pair $M = (E,\mathcal{F})$ is called a *simple matroid (on* E*)*; in particular it is a hereditary hypergraph, and we may consider for matroids the same concepts defined above for hypergraphs. In particular, the *rank* $r(S)$ will be defined by

$$r(S) = \max_{F\in\mathcal{F}} |F\cap S|.$$

Axiom (3) states that a member of the family $\mathcal{F}$ contained in S and maximal in S has cardinality $r(S)$.

(1) (P. Seymour, J.C.T., B23, 1977, 189-222). For a detailed exposition and for general terminology, we refer the reader to D. Welsh, *Matroid Theory*, Academic Press, New York 1976; R.E. Bixby, Matroids and Operations Research in H. Greenberg, F. Murphy, S. Shaw, *Advanced Techniques*, North Holland, Amsterdam, 1982.

In matroid theory, the elements of E are the *elements* of the matroid M, and the members of $\mathcal{F}$ are the *independent sets*. They are also the *edges* of the hypergraph $\mathcal{F}$. Those sets which do not appear in $\mathcal{F}$ are the *dependent sets*. A minimal dependent set is called a *circuit*.

Proposition 1. *If $M = (E,\mathcal{F})$ is a matroid of rank $r(E)$, then the maximal independent sets form a uniform hypergraph of rank $r(E)$.*

Clear.

Proposition 2. *If $M = (E,\mathcal{F})$ is a matroid of rank r, and if $A \subset E$, the subhypergraph $\mathcal{F}_A = \{F \cap A / F \in \mathcal{F}, F \cap A \neq \emptyset\}$ of M is a matroid of rank $r_A(S) = r(S)$.*

Clear.

Proposition 3. *If $M = (E,\mathcal{F})$ is a matroid, every k-section*

$$\mathcal{F}_{(k)} = \{F / 1 \leq |F| \leq k,\ F \in \mathcal{F}\}$$

forms a matroid of rank $r_{(k)}(S) = \min\{k, r(S)\}$.

Clear.

Example 1. The family $\mathcal{P}'(E)$ of non-empty subsets of a set E is a matroid of rank $r(S) = |S|$, and its strong stability number is $\overline{\alpha} = 1$.

The family $\mathcal{P}_{(k)}(E)$ of subsets of E of cardinality $\leq k$ and ≥ 1 is also a matroid, since it is the k-section of the preceding matroid. Its strong stability number is $\overline{\alpha} = 1$, its circuits are the subsets of E having $k+1$ elements.

Example 2. Take for E a finite set of vectors, and for $\mathcal{F}$ the family of linearly independent sets of vectors. Then $(E,\mathcal{F})$ is a matroid, and the rank $r(S)$ of a set S of vectors is the dimension of the linear space spanned by S; $\overline{\alpha}$ is the maximum number of vectors of E which are all colinear.

Example 3. Let G be a multigraph; take for E the edge-set of G, and for $\mathcal{F}$ the family of sets of edges which contain no cycles. $(E,\mathcal{F})$ is then a matroid with rank $r(S)$ equal to the cocyclomatic number of the partial graph generated by S. An independent set is a forest of G, and a circuit is an elementary cycle in G.

Example 4. Let G be a multigraph without bridges. Take for E the set of edges of

G, and for $\mathcal{F}$ the family of sets of edges of G whose suppression does not increase the number of connected components. $(E,\mathcal{F})$ is then a matroid, having rank $r(S)$ equal to the cyclomatic number of the partial graph generated by S. A base is a minimal co-forest, a circuit is an elementary cocycle of G.

Example 5 (Edmonds, Fulkerson 1965). Let G be a graph without isolated vertices and for every matching V denote by $S(V)$ the set of vertices saturated by the matching V; take as members of $\mathcal{F}$ every set F of vertices contained in at least one $S(V)$.

It can be shown that $(X,\mathcal{F})$ is a matroid of rank

$$r(S) = |S| - \max_{T\subset S}\{p_i(G_T) - |\Gamma_G(T) - T|\},$$

where $p_i(H)$ denotes the number of components of odd order in a subgraph H of G.

Example 6. For a family $(A_j/j\in Q)$ of subsets of a set E, set

$$A(Q) = \bigcup_{j\in Q} A_j = E;$$

we call a *partial transversal* a subset $T = \{t_1,t_2,\ldots,t_k\}$ of E such that there exists an injection $j(i)\colon\{1,2,\ldots,k\} \to Q$, with

$$t_i \in A_{j(i)} \qquad (i = 1,2,\ldots,k).$$

The family of partial transversals defines a matroid on E of rank

$$r(S) = |Q| + \min_{J\subset Q}(|A(J)\cap S| - |J|).$$

This matroid is called the *transversal matroid* of the family $\{A_j/j\in Q\}$.

Indeed, consider the bipartite graph (Q,E,Γ) formed by two sets $Q = \{1,2,\ldots,q\}$ and $E = \{x_1,x_2,\ldots,x_n\}$ where

$$\Gamma(j) = A_j \qquad (j\in Q).$$

We know from Example 5 that the family of sets of saturated vertices in a matching defines a matroid: the family of partial transversals is the trace on E of this matroid: it is thus a matroid. Its rank is given by König's Theorem:

$$r(S) = \min_{J\subset Q}(|Q-J| + |\Gamma(J)\cap S|) = q + \min_{J\subset Q}(|A(J)\cap S| - |J|).$$

Example 7. If $(C_1,C_2,\ldots,C_p)$ is a partition of a set E into p classes, and if $c_1,c_2,\ldots,c_p$ are integers with $1 \le c_i \le |C_i|$, the family

$$\mathcal{F} = \{F / F \subset E, F \neq \emptyset, |F \cap C_i| \leq c_i \text{ for each } i\}$$

defines a matroid on E of rank

$$r(S) = \sum_{i=1}^{p} \min\{c_i, |S \cap C_i|\}.$$

Example 8. Let G be a simple graph, and let k be an integer ≥ 2. A *k-star with centre x* is a partial graph of G formed by a set of $\leq k$ edges incident on x. Las Vergnas has shown that the sets S of vertices which may be covered by a family of pairwise vertex-disjoint k-stars form the independent sets of a matroid of rank

$$r(S) = \min_{T \subset S}\{k\,|\Gamma_G(T)| + |S - T|\}.$$

Example 9. Let f be a map from subsets of X to $\mathbb{N}$ such that

$$f(\emptyset) = 0$$

$$A \subset B \Rightarrow f(A) \leq f(B)$$

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

Edmonds, Rota, and Welsh showed that the sets S such that $|T| \leq f(T)$ for every $T \subset S$ form the independent sets of a matroid of rank

$$r(S) = \min_{T \supset S}\{f(T) + |S - T|\}.$$

We shall now prove two propositions which we will need for the following.

Proposition 4. *If $M = (E, \mathcal{F})$ is a matroid, then its rank $r(A)$ satisfies the following properties:*

(1) $\qquad r(\emptyset) = 0,$

(2) $\qquad r(\{e\}) = 1 \quad (e \in E)$

(3) $\qquad A \subset B \Rightarrow r(A) \leq r(B),$

(4) $$r(A) + r(B) \geq r(A \cup B) + r(A \cap B).$$

Properties (1), (2) and (3) are clear. We shall prove (4). Let F be an independent set contained in $A \cap B$ with $|F| = r(A \cap B)$.

Let F_A be an independent set contained in A with $|F_A| = r(A)$ and $F_A \supset F$.

Let E_0 be an independent set containing F_A, contained in $A \cup B$, with $|E_0| = r(A \cup B)$.

Clearly $E_0 \cap A = F_A$ (since F_A is a maximal independent set in A) and $E_0 \cap (A \cap B) = F$ (since F is a maximal independent set in $A \cap B$). Then

$$\begin{aligned} r(A \cup B) &= |E_0| + |(E_0 \cap A) \cup (E_0 \cap B)| \\ &= |E_0 \cap A| + |E_0 \cap B| - |E_0 \cap A \cap B| \\ &\leq |F_A| + r(B) - |F| = r(A) + r(B) - r(A \cap B). \end{aligned}$$

Thus (4) follows.

(Properties (1), (2), (3), (4) are characteristics of the rank and may also be taken as the axioms of a matroid on E).

Proposition 5. *If, in a matroid M, we have $F \in \mathcal{F}$ and $F \cup \{a\} \in \mathcal{F}$, then the set $F \cup \{a\}$ contains exactly one circuit.*

Let F be a minimum independent set which would be a counterexample. Since $F \cup \{a\}$ contains two distinct circuits C_1 and C_2, we have $a \in C_1$, $a \in C_2$. By the minimality of C_1 and C_2 there exists a point $a_1 \in C_1 - C_2$, and a point $a_2 \in C_2 - C_1$.

1. The set $A_0 = F \cup \{a\} - \{a_1, a_2\}$ is independent. Otherwise, consider the set $F' = F - \{a\}$, which is independent as it is contained in F. The set $F' \cup \{a\}$ contains the circuit C_2 and a minimal dependent set of A_0; thus it contains two distinct circuits, and as $|F'| < |F|$, this contradicts the minimality of F.

2. The submatroid spanned by $F \cup \{a\}$ is a matroid of rank $|F|$ which contains the independent set A_0. Since $|A_0| < |F|$ we have

$$A_0 \cup \{a_i\} \in \mathcal{F}$$

for $i = 1,2$. The contradiction follows, as C_i is a dependent set contained in $A_0 \cup \{a_i\}$.

Lemma. *If S is a maximal strongly stable set in a matroid $(E, \mathcal{F})$, then each $s \in S$ is adjacent to every $a \in E - S$.*

Consider a maximal strongly stable set S. Then $r(S) = 1$. Let

$$S = \{s_1, s_2, \ldots, s_p\}.$$

Consider a point $a \in E - S$; since $S \cup \{a\}$ is not strongly stable there exists an $s_j \in S$ adjacent to a. If $k \neq j$ the vertex $s_k \in S$ is adjacent to $\{a, s_j\}$, since the set $A = \{a, s_j, s_k\}$ is of rank 2, and an independent set containing s_k is contained in a maximal independent set F satisfying $|F \cap A| = 2$. Thus a is adjacent to s_k, for every k.

Theorem 1. *If $M = (E, \mathcal{F})$ is a matroid with strong stability number $\overline{\alpha}(M) \geq \frac{|E|}{2}$ then $\overline{\alpha}(M) = \rho(M)$.*

Indeed, consider a maximum strongly stable set

$$S = \{s_1, s_2, \ldots, s_p\};$$

we may write

$$E - S = \{a_1, a_2, \ldots, a_q\}, \quad q \leq p.$$

From the lemma, there exists an edge F_{ij} which contains a_i and s_j, and E can be covered by the p edges $F_{1,1}, F_{2,2}, \ldots, F_{q,q}, F_{q,q+1}, \ldots, F_{q,p}$; thus

$$\rho(M) \leq p = \overline{\alpha}(M).$$

Since the reverse inequality also holds, we have $\rho(M) = \overline{\alpha}(M)$.

Theorem 2 *A matroid $M = (E, \mathcal{F})$ is conformal if and only if there exists a partition $(S_1, S_2, \ldots, S_q)$ of E such that $\mathcal{F}$ consists of the family of non-empty sets F with*

$$|F \cap S_i| \leq 1 \qquad (i = 1, 2, \ldots, q).$$

Let $S_1 = \{s_1, s_2, \ldots, s_p\}$ be a maximal strongly stable set in a conformal matroid of rank $h = r(E)$. It is sufficient to show that the family $\mathcal{F}$ is of the desired form.

1. Let F_1 be a maximal independent set containing the point s_1. Put

$$A = E - S_1$$

$$A_1 = F_1 \cap A.$$

Then $|F_1| = h$, so $|A_1| = h - 1$.

2. We shall show that A_1 is a maximal independent set in A. Indeed, if this were not the case, there would exist an $a \in A$ with

$A_1 \cup \{a\} \in \mathcal{F}$.

From the lemma, the vertices a and s_1 are adjacent and are thus contained in a maximal independent set F_{a,s_1}. As the matroid M is conformal, from Theorem 15, Chapter 1, there exists an $F_0 \in \mathcal{F}$ such that

$$F_0 \supset [F \cap (A_1 \cup \{a\})] \cup [A_1 \cup \{a\}) \cap F_{a,s_1}] \cup (F_{a,s_1} \cap F)$$
$$= A_1 \cup \{a\} \cup \{s_1\}$$

Thus $|F_0| \geq h + 1$ contradicting that h is the rank of M.

3. From the above, we have $r(A) = h - 1$, so every maximal independent set F satisfies

$|F \cap S_1| = 1$.

In the submatroid induced by A, which is of rank $h-1$, consider a maximal strongly stable set S_2; as above we see that

$|F \cap S_2| = 1$.

We determine thus a partition $S_1, S_2, \ldots, S_h$ of E and every maximal independent set F of M satisfies $|F \cap S_i| = 1$ for $i = 1, 2, \ldots, h$.

4. Conversely, every set F which satisfies the above equalities has its points pairwise adjacent, and since the matroid is conformal and of rank h it is a maximal independent set.

The family $\mathcal{F}$ is thus of the desired form.

Q.E.D.

Let $\mathcal{A} = (A_1, A_2, \ldots, A_q) = (A_i / q \in Q)$ be a family of subsets of a set E. A *family of distinct representatives* is a family $(a(i)/i \in Q)$ of elements of E such that

(1) $\quad i \neq j \Rightarrow a(i) \neq a(j)$

(2) $\quad a(i) \in A_i \quad (i = 1, 2, \ldots, q)$.

The point $a(i)$ is the *representative* of the set A_i. Clearly a family of distinct representatives defines a transversal of cardinality q; the converse, however, is not true.

If we consider the bipartite graph (Q,E,Γ) with $e \in \Gamma(i)$ if $e \in A_i$, a set of distinct representatives is the image of a matching of Q into E.

If $J \subset Q$, put $A(J) = \bigcup_{j \in J} A_j$; a necessary and sufficient condition for the existence of a family of distinct representatives, from König's theorem, is that

$$|A(J)| \geq |J| \qquad (J \subset Q).$$

The following theorems are generalisations of this result.

Theorem 3 (Perfect, 1969). *Let $M = (E,\mathcal{F})$ be a matroid of rank $r(E)$, let k be an integer $\leq r(E)$ and let $\mathcal{A} = (A_1,A_2,\ldots,A_q) = (A_i/i \in Q)$ be a family of q subsets of E; a necessary and sufficient condition for the existence of an independent set $F = \{a(i)/i \in K\}$, $K \subset Q$, $|K| = k$, with $a(i) \in A_i$ for every $i \in K$ is that we have*

$$r(A(J)) \geq |J| + k - q \qquad (J \subset Q).$$

1. If there exists such an independent set F, we have

$$r(A(J)) \geq |F \cap A(J)| \geq |K \cap J| = |K| + |J| - |K \cup J|$$
$$\geq k + |J| - q$$

Thus we have the stated inequality.

2. Conversely, suppose the inequality holds. Consider the family $\mathcal{B} = (B_i/i \in Q)$ with

$$(1) \qquad \begin{cases} B_i \subset A_i & (i \in Q) \\ r(B(J)) \geq |J| + k - q & (J \subset Q) \end{cases}$$

The relation $\mathcal{B} \prec \mathcal{B}'$ meaning $B_i \subset B_i'$ for every $i \in Q$ is an order relation. Consider a family $\mathcal{B} = (B_1,B_2,\ldots,B_q)$ which is minimal with respect to this order. We shall show that $|B_i| = 1$ for every i.

Indeed, if for example $|B_1| > 1$, there exist two points $b',b'' \in B_1$ with $b' \neq b''$. Put

$$B_1' = B_1 - \{b'\}$$
$$B_1'' = B_1 - \{b''\}$$
$$B_i' = B_i'' = B_i \ \text{ if } \ i \neq 1.$$

By the minimality of B there exist two subsets $I,J \subset Q$ with

$$r(B'(I)) < |I| + k - q$$

$$r(B''(I)) < |J| + k - q$$

Thus

$$r(B'(I)) + r(B''(J)) \leq |I| + |J| + 2(k-q) - 2.$$

Further,

$$B'(I) \cup B''(J) = B(I \cup J)$$

$$B'(I) \cap B''(J) = B(I \cap J - \{1\}).$$

From proposition 4 we may write

$$\begin{aligned} r(B'(I)) + r(B''(J)) &\geq r(B'(I) \cup B''(J)) + r(B'(I) \cap B''(J)) \\ &\geq r(B(I \cup J)) + r(B(I \cap J - \{1\})) \\ &\geq |I \cup J| + |I \cap J - \{1\}| + 2(k-q) \\ &\geq |I| + |J| - 2(k-q) - 1. \end{aligned}$$

A contradiction follows.

We have thus shown that $\mathcal{B}$ is of the form $(\{b_i\}/i \in Q)$. Put

$$B = \{b_i/i \in Q\}$$

From (1) we have

$$r(B) = r(B(Q)) \geq |Q| + k - q = k.$$

Thus there exists a set $K \subset Q$ with $|K| = k$, and an independent set $F = \{b_i/i \in K\} \subset B$ with

$$b_i \in A_i \qquad (i \in K).$$

Q.E.D.

As an immediate consequence, we have the well known theorem of Rado:

Theorem 4 (Rado, 1942). *If $M = (E, \mathcal{F})$ is a matroid, a family $\mathcal{A} = (A_1, \ldots, A_q)$ of subsets of E has an independent set of distinct representatives if and only if*

$$r(A(J)) \geq |J| \qquad (J \subset Q).$$

Indeed, let $k = q$ in the statement of Theorem 3.

Corollary 1. *Two families $\mathcal{A} = (A_1, A_2, \ldots, A_q)$ and $\mathcal{B} = (B_1, B_2, \ldots, B_q)$ have a common* *set of distinct representatives if and only if*

$$|A(J) \cap B(K)| \geq |J| + |K| - q \qquad (J,K \subset Q).$$

Indeed, consider the transversal matroid M of the family $\mathcal{B}$ (example 6), whose rank is

$$r(S) = q + \min_{K \subset Q}(|B(K) \cap S| - |K|).$$

There exists a transversal set of $\mathcal{A}$ which is independent in M if and only if, for every $J \subset Q$, we have

$$r(A(J)) = q + \min_{K \subset Q}(|A(J) \cap B(K)| - |K|) \geq |J|$$

giving us the desired condition.

Corollary 2. *If* $\mathcal{C} = (C_1, C_2, \ldots, C_p)$ *is a partition of* E, *and if* $c_1, c_2, \ldots, c_p$ *are integers with* $0 \leq c_i \leq |C_i|$ *for each* i, *a family* $\mathcal{A} = (A_1, A_2, \ldots, A_q)$ *has a set* T *of distinct representatives with* $|T \cap C_i| \leq c_i$ *for every* i *if and only if*

$$\sum_{i=1}^{p} \min\{c_i, |A(J) \cap C_i|\} \geq |J| \qquad (J \subset Q).$$

Indeed, consider the matroid M formed by the sets $F \subset E$ with $|F \cap C_i| \leq c_i$ for each i (example 7), whose rank is

$$r(S) = \sum_{i=1}^{p} \min\{c_i, |S \cap C_i|\}.$$

There exists a set of distinct representatives of $\mathcal{A}$ which is independent in M if and only if, for every $J \subset Q$,

$$r(A(J)) = \sum_{i=1}^{p} \min\{c_i, |A(J) \cap C_i|\} \geq |J|$$

Q.E.D.

Recall the proposition (cf. *Graphs*, Corollary to Theorem 6, Chap. 7) which says: A necessary and sufficient condition for a set $B \subset E$ to be contained in a set of distinct representatives of a family $\mathcal{A} = (A_1, A_2, \ldots, A_q)$ is that

$$\min\{|A(J) \cup B|, q - |B - A(J)|\} \geq |J| \qquad (J \subset Q).$$

This may be extended to matroids; first we shall prove a lemma:

Lemma. *Let $M = (E,\mathcal{F})$ be a matroid of rank r, let $B \in \mathcal{F}$, and let $q \geq |B|$; the family*

$$\mathcal{F}_{B,q} = \{F/F \subset E, F \cup B \in \mathcal{F}, |F \cup B| \leq q\}$$

defines a matroid on E and its rank is

$$r_{B,q}(S) = \min\{r(S \cup B), q\} - |B - S|.$$

Let $S \subset E$, and let S_0 be a subset of S that belongs to $\mathcal{F}_{B,q}$: every set F with

(1) $\qquad F \in \mathcal{F}_{B,q}$

(2) $\qquad S_0 \subset F \subset S$

clearly satisfies

$$|F| \leq \min\{r(S \cup B), q\} - |B - S|.$$

It thus remains to show that equality can hold.

The set $B \cup S_0$, being independent in M, is contained in an independent set F' of $B \cup S$ with

$$|F'| = r(S \cup B).$$

Let F'' be an independent set with

$$B \cup S_0 \subset F'' \subset F'; \quad |F''| = \min\{r(S \cup B), q\}.$$

The set $F = F'' \cap S$ satisfies (1) and (2), and

$$|F| = \min\{r(S \cup B), q\} - |B - S|.$$

Q.E.D.

Theorem 5 (Las Vergnas, 1969). *Let $M = (E,\mathcal{F})$ be a matroid of rank r, and let $B \in \mathcal{F}$, and $q \geq |B|$; a family $\mathcal{A} = (A_1, A_2, \ldots, A_q)$ of subsets of E has a family of distinct representatives which is an independent set containing B if and only if*

$$\min\{r(A(J) \cup B), q\} - |A(J) - B| \geq |J| \quad (J \subset Q).$$

Consider the matroid on E defined by the family

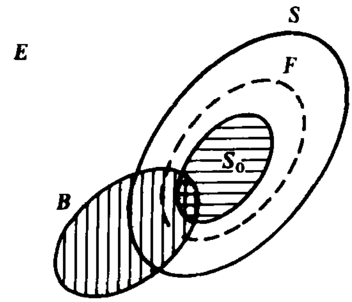


Figure 21

$$\mathcal{F}_{B,q} = \{F/F \subset E, F \cup B \in \mathcal{F},\ |F \cup B| \leq q\}.$$

If there exists a set T of distinct representatives of $\mathcal{A}$ with $T \in \mathcal{F}$, $T \supset B$, then $|T| = q$, so

$$T \in \mathcal{F}_{B,q}.$$

Conversely, if there exists a set T of distinct representatives of $\mathcal{A}$ with $T \in \mathcal{F}_{B,q}$, then

$$T \cup B \in \mathcal{F},\quad |T \cup B| \leq q,\quad |T| = q,$$

so

$$T \in \mathcal{F};\ \ T \supset B.$$

Thus, from Rado's theorem, there exists a set T satisfying the conditions of the statement if and only if the rank $r_{B,q}$ of the matroid $(E,\mathcal{F}_{B,q})$ satisfies

$$r_{B,q}(A(J)) \geq |J| \qquad (J \subset Q)$$

or

$$\min\{r(A(J) \cup B), q\} - |B - A(J)| \geq |J|.$$

Q.E.D.

Let $M = (E,\mathcal{F})$ be a matroid on $E = \{e_1, e_2, \ldots, e_m\}$ and consider a map ϕ of E onto $\overline{E}$; define *the image of M under* ϕ to be the hypergraph

$$\overline{\mathcal{F}} = (\phi(F)/F \in \mathcal{F}).$$

As ϕ is a map *onto* $\overline{E}$, $\overline{M} = (\overline{E},\overline{\mathcal{F}})$ is clearly a hypergraph, which we shall now study.

Theorem 6 (Nash-Williams, 1966). *If* $\overline{M} = (\overline{E},\overline{\mathcal{F}})$ *is the image of a matroid* $M = (E,\mathcal{F})$ *under a map* ϕ *of* E *onto* $\overline{E}$, *then* $\overline{M}$ *is a matroid and its rank is*

$$\bar{r}(\overline{E}) = \min_{\overline{A} \subset \overline{E}}(r(\phi^{-1}(\overline{A})) + |\overline{E}-\overline{A}|)$$

1. We shall show that:

$$\max_{\overline{F}} |\overline{F}| = \min_{\overline{A} \subset \overline{E}}(r(\phi^{-1}(A)) + |\overline{E}-\overline{A}|).$$

Clearly, $\max |\overline{F}|$ is the greatest integer k such that the family

$$(\phi^{-1}(\bar{e}_1),\phi^{-1}(\bar{e}_2), \ldots, \phi^{-1}(\bar{e}_m))$$

has a partial set of distinct representatives that is an independent set in M and has cardinality k. From Theorem 3, this is the greatest integer k such that

$$\min_{\overline{A} \subset \overline{E}}(r(\phi^{-1}(\overline{A})) + |\overline{E}-\overline{A}|) \geq k,$$

whence

$$\max |\overline{F}| = \min_{\overline{A} \subset \overline{E}}(r(\phi^{-1}(\overline{A})) + |\overline{E}-\overline{A}|).$$

2. Now, it remains to show that the image of M under ϕ is a matroid.

Consider a map ϕ of $E = \{e_1,e_2,\ldots,e_m\}$ onto $\overline{E} = \{\bar{e}_2,\bar{e}_3, \ldots, \bar{e}_m\}$ satisfying

$$\phi(e_1) = \bar{e}_2$$

$$\phi(e_i) = \bar{e}_i \quad \text{if } i \neq 1.$$

We shall say that ϕ is a map *contracting* the set $\{e_1,e_2\}$; since every map is a composition of contracting maps, it suffices to show that the image of the matroid M under the contracting map ϕ is a matroid. Consider an independent set $F_0 \in \mathcal{F}$ such that $\overline{F}_0$ is maximal in $\overline{\mathcal{F}}$; we shall show that $\overline{F}_0$ is maximum for $\overline{\mathcal{F}}$; that is, from Part 1, that there exists a set $\overline{A} \subset \overline{E}$ with

$$|\overline{F}_0| = r(\phi^{-1}(\overline{A})) + |\overline{E}-\overline{A}|.$$

For simplicity, set $E_0 = E - \{e_1,e_2\}$. The set $\overline{F}_0$ being maximal in $\overline{\mathcal{F}}$ we may suppose that F_0 is a maximum set of $\mathcal{F}$.

We may also suppose that F_0 contains both e_1 and e_2, since otherwise $\overline{F}_0$ will be maximum, as we may write:

$$|\overline{F}_0| = |F_0| = r(E) = r(\phi^{-1}(\overline{E})) + |\overline{E}-\overline{E}|.$$

We shall distinguish three cases.

Case 1: $r(E_0) = r(E)$.

Since

$$r(E_0) \leq r(E-\{e_1\}) \leq r(E),$$

we also have $r(E-\{e_1\}) = r(E)$. As F_0 contains e_1 and e_2, we have

$$|F_0-\{e_1\}| = r(E)-1 < r(E-\{e_1\}).$$

Consequently there exists a maximum independent set F'_0 which does not contain the point e_1 and satisfies

$$\overline{F'_0} \supset \overline{F_0-\{e_1\}} = \overline{F}_0.$$

Thus $\overline{F}_0 = \overline{F'_0}$ and may write:

$$|\overline{F}_0| = |\overline{F'_0}| = |F'_0| = r(E) = r(\phi^{-1}(\overline{E}))+|\overline{E}-\overline{E}|.$$

Case 2: $r(E_0) = r(E) - 1$. Every maximum independent set thus contains e_1 or e_2. Further, we have

$$|F_0 \cap E_0| = |F_0| - 2 = r(E) - 2 < r(E_0).$$

Thus there exists a point $a \in E_0$ such that $(F_0 \cap E_0) \cup \{a\}$ is an independent set of cardinality $r(E_0) = r(E)-1$; let F'_0 be a maximum independent set which contains this set. Since F'_0 is maximum, it contains e_1 or e_2, for example:

$$F'_0 = (F_0 \cap E_0) \cup \{a,e_1\}.$$

Clearly $\overline{F'_0} \supset \overline{F}_0$, so $\overline{F}_0 = \overline{F'_0}$ and we may write:

$$|\overline{F'_0}| = |\overline{F}_0| = r(E) = r(\phi^{-1}(\overline{E})) + |\overline{E}-\overline{E}|.$$

Case 3: $r(E_0) = r(E)-2$. Every maximum independent set thus contains both e_1 and e_2. We have $\overline{F}_0 \supset \overline{F}$, so

$$|\bar{F}_0| = |F_0|-1 = r(E)-1$$
$$= r(E_0)+1 + r(\phi^{-1}(\bar{E}_0))+|\bar{E}-\bar{E}_0|.$$

In each of these cases, the set $\bar{F}_0$ is maximum for $\bar{\mathcal{F}}$.

Q.E.D.

Let $H^1,H^2, \ldots ,H^p$ be hypergraphs on a set X of vertices; their *join* is the hypergraph

$$H = H^1 \vee H^2 \vee \cdots \vee H^p$$

defined by the family

$$H = \{E^1 \cup E^2 \cup \cdots \cup E^p / E^1 \in H^1, E^2 \in H^2, \ldots , E^p \in H^p\}.$$

H is clearly a hypergraph on X.

Theorem 7. *If* $(E,\mathcal{F}^1),(E,\mathcal{F}^2,...,(E,\mathcal{F}^p)$ *are matroids of rank* $r^1,r^2, \ldots ,r^p$ *respectively, their hypergraph-join is a matroid of rank*

$$\bar{r}(E) = \min_{A \subset E}(r^1(A)+ \cdots +r^p(A)+|E-A|).$$

Make p identical copies $E^1,E^2,...,E^p$ of the set E, and consider the map ϕ of $\bigcup_{i=1}^{p} E^i$ into E which maps each $e_k^i \in E^i$ to the corresponding $e_k \in E$. $M = (\cup E^i, \vee \mathcal{F}^i)$ is clearly a matroid, and its rank is

$$r(X) = r^1(E^1)+r^2(E^2)+...+r^p(E^p).$$

From Theorem 6, the image of this matroid under the map ϕ is also a matroid $\overline{M}$, which is exactly the join $\bigvee_{i=1}^{p} \mathcal{F}_i^i$; the rank of the matroid-join $\overline{M}$ is thus

$$\bar{r}(E) = \min_{A \subset E}(r(\phi^{-1}(A))+|E-A|)$$
$$= \min_{A \subset E}(\sum_{i=1}^{p} r^i(A)+|E-A|).$$

Corollary 1 (Edmonds, 1968; Nash-Williams, 1968). *For a matroid* $M = (E,\mathcal{F})$ *the minimum number of independent sets required to cover* E *is*

$$\rho(M) = \max_{\substack{A \subset E \\ A \neq \emptyset}} \left[\frac{|A|}{r(A)} \right]^* .$$

By definition, $\rho(M)$ is the least integer k such that the join $M \vee M \vee \cdots \vee M$ of k matroids identical to M is of rank $|E|$, or, from Theorem 7, the least integer k such that

$$\min_{A \subset E} (kr(A) + |E - A|) = |E|.$$

This is equivalent to:

$$\min_{A \subset E} (kr(A) - |A|) = 0,$$

or

$$kr(A) - |A| \geq 0 \qquad (A \subset E),$$

or

$$k \geq \frac{|A|}{r(A)} \qquad (A \subset E,\ A \neq \emptyset).$$

We thus have

$$\rho(M) = \max_{\substack{A \subset E \\ A \neq \emptyset}} \left[\frac{|A|}{r(A)} \right]^* .$$

Corollary 2. *If $M = (E, \mathcal{F})$ is a matroid, the maximum number k_0 of maximal independent pairwise disjoint sets is*

$$k_0 = \min_{\substack{A \subset E \\ r(A) \neq r(E)}} \left[\frac{|E - A|}{r(E) - r(A)} \right].$$

Indeed, k_0 is the largest integer k such that the matroid-join of k matroids identical to M is of rank $kr(E)$, or

$$\min_{A \subset E} (kr(A) + |E - A|) = kr(E).$$

This is equivalent to

$$\min_{A \subset E} (kr(A) - kr(E) + |E - A|) = 0,$$

or

$$k(r(E)-r(A)) \leq |E-A| \quad (A \subset E)$$

giving us the stated formula.

Corollary 3. *Consider a matroid* $M = (E, \mathcal{F})$ *and a sequence* $k_1, k_2, \ldots, k_q$ *with*

$$r(E) \geq k_1 \geq k_2 \geq \cdots \geq k_q > 0; \quad \sum_{i=1}^{q} k_i = |E|.$$

Let k_j^* *be the number of* k_i*'s which are* $\geq j$. *The set* E *can be partitioned into* q *independent sets* $F_1, F_2, \ldots, F_q$ *with* $|F_i| = k_i$ *for each* i *if and only if*

$$\sum_{j > r(A)} k_j^* \geq |E-A| \quad (A \subset E).$$

Consider the k_i-section $M_{(k_i)}$, defined by the family

$$\mathcal{F}_{(k_i)} = \{F / F \in \mathcal{F},\ |F| \leq k_i\}.$$

This is a matroid of rank $r^i(A) = \min\{r(A), k_i\}$, and the matroid-join

$$M = \bigvee_{i=1}^{q} M_{(k_i)}$$

is of rank $|E|$. Thus

$$\min_{A \subset E} \Big(\sum_{i=1}^{q} r^i(A) + |E-A|\Big) = |E|.$$

This is equivalent to

$$\sum_{i=1}^{q} \min\{r(A), k_i\} + |E-A| \geq |E| = \sum_{i=1}^{q} k_i = \sum_{j>0} k^* \quad (A \subset E).$$

Hence

$$\sum_{j>0} k_i^* - \sum_{j=1}^{r(A)} k_j^* \leq |E-A| \quad (A \subset E)$$

giving us the stated condition.

Corollary 4. *The chromatic number of the hypergraph* H^M *consisting of the circuits of a matroid* $M = (E, \mathcal{F})$ *of rank* r *is equal to*

$$\chi(H^M) = \max_{\substack{A \subset E \\ A \neq \emptyset}} \left[\frac{|A|}{r(A)}\right]^* .$$

Indeed, a set $F \subset E$ is independent if and only if it contains no circuits. Thus a partition $(A_1,A_2,...,A_q)$ is a colouring of the hypergraph H^M if and only if $A_1,A_2,...,A_q$ are independent sets; thus $\chi(H^M) = \rho(M)$ and corollary 1 gives the stated formula.

The preceding results allow us to obtain rapidly some results for graphs, originally proven by direct but much longer methods.

Application 1 (Tutte, 1961). *The set of edges of a simple connected graph $G = (X,E)$ contains k pairwise disjoint spanning trees if and only if, for every partition $\mathcal{P}$ of X, the number of edges $m_G(\mathcal{P})$ which join vertices in distinct classes of the partition satisfies*

$$m_G(\mathcal{P}) \geq k(|\mathcal{P}|-1).$$

1. If there exist k spanning trees $H_1,...,H_k$ in G, pairwise edge-disjoint, then for a partition $\mathcal{P}$ of the vertices,

$$m_{H_i}(\mathcal{P}) \geq |\mathcal{P}|-1 \qquad (i=1,2,...,k).$$

Thus

$$m_G(\mathcal{P}) \geq \sum_{i=1}^{k} m_{H_i}(\mathcal{P}) \geq k(|\mathcal{P}|-1).$$

2. If the stated condition holds, consider the matroid $M = (E,\mathcal{F})$ on E defined by the family of forests $F_1,F_2,...,F_q \subset E$ of the graph G; if $r(A)$ denotes the rank of M and if $A \subset E$ defines a partial graph of G having p connected components, forming a partition

$$\mathcal{P} = (X_1,X_2,...,X_p)$$

of X, we have

$$r(E)-r(A) = (n-1)-(n-p) = p-1 = |\mathcal{P}|-1.$$

Thus, from the conditions in the statement, we have

$$|E-A| \geq m_G(\mathcal{P}) \geq k(|\mathcal{P}|-1) = k(r(E)-r(A)).$$

Hence,

$$k \leq \min_{\substack{A \subset E \\ r(A) \neq r(E)}} \left(\frac{|E-A|}{r(E)-r(A)} \right).$$

Thus, from Corollary 2 to Theorem 7, there exist k disjoint spanning trees in G.

Application 2 (Nash-Williams, 1964). *The edges of a simple graph $G = (X,E)$ may be coloured with k colours in such a way that no cycle is monochromatic if and only if for every set $A \subset X$ the number $m_G(A,A)$ of edges having both ends in A satisfies*

$$m_G(A,A) \leq k(|A|-1).$$

In other words, the chromatic number of the hypergraph G^C formed by the cycles of edges of G is equal to

$$\chi(G^C) = \max_{|A|>1} \left[\frac{m_G(A,A)}{|A|-1} \right]^*$$

1. If the edges of G are coloured thus with k colours $1,2,\ldots,k$, let $m_i(A,A)$ be the number of edges of colour i having both ends in A; since these edges form a forest, $m_i(A,A) \leq |A|-1$. Thus

$$m_G(A,A) = m_1(A,A)+\ldots+m_k(A,A) \leq k(|A|-1)$$

as stated.

2. Conversely, suppose that the condition of the Theorem is satisfied. Consider the matroid $(E,\mathcal{F})$ formed by the family of forests of the graph G, and let r be its rank. If the partial graph (X,F) of G generated by $F \subset E$ has p connected components $(X_1,F_1),(X_2,F_2),\ldots,(X_p,F_p)$ which are not isolated points, then

$$kr(F_i)-|F_i| \geq k(|X_i|-1)-m_G(X_i,X_i) \geq 0.$$

Hence

$$kr(F)-|F| = \sum_{i=1}^{p}(kr(F_i)-|F_i|) \geq 0,$$

or

$$k \geq \max_{\substack{F\subset E \\ F\neq\emptyset}} \left[\frac{|F|}{r(F)} \right]^*.$$

Thus, from Corollary 4, it is possible to colour the edges of G with k colours so that no cycle is monochromatic.

Application 3. *If G is a simple graph of maximum degree h, it is possible to colour its edges with $[\frac{h}{2}]^*+1$ colours so that no cycle is monochromatic.*

Indeed, let $G = (X,E)$ and let $A \subset X$. Suppose $|A| > 1$ and $a \in A$, and set $\overline{A} = A - \{a\}$. Then

$$\frac{1}{|A|-1} m_G(A,A) = \frac{1}{|\overline{A}|}(m_G(\overline{A},\overline{A}) + m_G(\overline{A},a))$$
$$\leq \frac{1}{|\overline{A}|}(|\overline{A}|\frac{h}{2} + |\overline{A}|) \leq \frac{h}{2}+1.$$

From Nash-Williams' Theorem (Application 2), it follows that:

$$\chi(G^C) \leq [\frac{h}{2}]^* + 1.$$

Thus it is possible to colour the edges of G with $[\frac{h}{2}]^*+1$ colours so that no cycle of G is monochromatic.

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