Chapter 4

Set Theory

"A set is a Many that allows itself to be thought of as a One."
(Georg Cantor)

In the previous chapters, we have often encountered "sets", for example, prime numbers form a set, domains in predicate logic form sets as well. Defining a set formally is a pretty delicate matter, for now, we will be happy to consider an intuitive definition, namely:

Definition 24. A set is a collection of abstract objects.

A set is typically determined by its distinct elements, or members, by which we mean that the order does not matter, and if an element is repeated several times, we only care about one instance of the element. We typically use the bracket notation $\{\}$ to refer to a set.

Example 42. The sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same, because the ordering does not matter. The set $\{1, 1, 1, 2, 3, 3, 3\}$ is also the same set as $\{1, 2, 3\}$, because we are not interested in repetition: either an element is in the set, or it is not, but we do not count how many times it appears.

One may specify a set *explicitly*, that is by listing all the elements the set contains, or *implicitly*, using a predicate description as seen in predicate logic, of the form $\{x,\ P(x)\}$. Implicit descriptions tend to be preferred for infinite sets.

Example 43. The set A given by $A = \{1, 2\}$ is an explicit description. The set $\{x,\ x \text{ is a prime number }\}$ is implicit.

Set

A set is a collection of abstract objects

- Examples: prime numbers, domain in predicate logic

- Determined by (distinct) elements/members.
  - E.g. {1, 2, 3} = {3, 1, 2} = {1, 3, 2}= {1, 1, 1, 2, 3, 3, 3}

- Two common ways to specify a set
  - Explicit: Enumerate the members

    e.g. A= {2, 3}
  - Implicit: Description using predicates $\{x \mid P(x)\}$

    e.g. A = {x| x is a prime number}

Membership & Subset

We write $x \in S$ iff x is an element (member) of S.

- e.g. A = {x| x is a prime number} then $2 \in A$, $3 \in A$, $5 \in A, \ldots, 1 \notin A$, $4 \notin A$, $6 \notin A$, …

A set *A* is a *subset* of the set *B*, *denoted by* $A \subseteq B$ iff every element of *A* is also an element of *B*. i.e.,

- $A \subseteq B \triangleq \forall x(x \in A \rightarrow x \in B)$
- $A \not\subseteq B \triangleq \neg (A \subseteq B)$

  $\equiv \neg \forall x(x \in A \rightarrow x \in B)$

  $\equiv \exists x(x \in A \land x \notin B)$

Subset versus Membership: S = {rock, paper, scissors}

R = {rock}, $R \subseteq S$, rock $\in S$

Given a set S, one may be interested in elements belonging to S, or in subset of S. The two concepts are related, but different.

Definition 25. A set A is a subset of a set B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B. Formally

$$A \subseteq B \iff \forall x(x \in A \to x \in B).$$

Note the two notations $A \subset B$ and $A \subseteq B$: the first one says that A is a subset of B, while the second emphasizes that A is a subset of B, possibly equal to B. The second notation is typically preferred if one wants to emphasize that one set is possibly equal to the other.

To say that A is not a subset of S, we use the negation of $\forall x(x \in A \to x \in B)$, which is (using the rules we have studied in predicate logic! namely negation of universal quantifier, conversion theorem, and De Morgan's law) $\exists x(x \in A \wedge x \notin B)$. The notation is $A \not\subseteq B$.

For an element x to be an element of a set S, we write $x \in S$. This is a notation that we used already in predicate logic. Note the difference between $x \in S$ and $\{x\} \subseteq S$: in the first expression, x is in element of S, while in the second, we consider the subset $\{x\}$, which is emphasized by the bracket notation.

Example 44. Consider the set $S = \{$ rock, paper, scissors $\}$, then $R = \{$ rock $\}$ is a subset of S, while rock $\in S$, it is an element of S.

Definition 26. The empty set is a set that contains no element. We denote it $\varnothing$ or $\{\}$.

There is a difference between $\varnothing$ and $\{\varnothing\}$: the first one is an empty set, the second one is a set, which is not empty since it contains one element: the empty set!

Definition 27. The empty set is a set that contains no element. We denote it $\varnothing$ or $\{\}$.

Example 45. We say that two sets A and B are equal, denoted by $A = B$, if and only if $\forall x, (x \in A \leftrightarrow x \in B)$.

To say that two sets A and B are not equal, we use the negation from predicate logic, which is:

$$\neg(\forall x, (x \in A \leftrightarrow x \in B)) \equiv \exists x((x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)).$$

Empty Set

The set that contains no element is called the ***empty set*** or ***null set***.

- The empty set is denoted by $\varnothing$ or by { }.
- Note: $\varnothing \neq \{\varnothing\}$

Set Equality

$$A=B \triangleq \forall x(x\in A \leftrightarrow x\in B)$$

- Two sets A, B are equal *iff* they have the same elements.

$$A\neq B \triangleq \neg\forall x(x\in A \leftrightarrow x\in B)$$
$$\equiv \exists x\,[(x\in A\wedge x\notin B) \vee (x\in B\wedge x\notin A)]$$

- Two sets are not equal if they do not have identical members, i.e., there is some element in one of the sets which is absent in the other.

- Example:
  {1, 2, 3} = {3, 1, 2} = {1, 3, 2}= {1, 1, 1, 2, 3, 3, 3}

This makes our earlier example $\{1,2,3\} = \{1,1,1,2,3,3,3\}$ easier to justify than what we had intuitively before: both sets are equal because whenever a number belongs to one, it belongs to the other.

Definition 28. The cardinality of a set S is the number of distinct elements of S. If $|S|$ is finite, the set is said to be finite. It is said to be infinite otherwise.

We could say the number of elements of S, but then this may be confusing when elements are repeated as in $\{1,2,3\} = \{1,1,1,2,3,3,3\}$, while there is no ambiguity for distinct elements. There $|S| = |\{1,2,3\}| = 3$. The set of prime numbers is infinite, while the set of even prime numbers is finite, because it contains only 2.

Definition 29. The power set $P(S)$ of a set S is the set of all subsets of S:

$$P(S) = \{A,\ A \subseteq S\}.$$

If $S = \{1,2,3\}$, then $P(S)$ contains S and the empty set $\varnothing$, and all subsets of size 1, namely $\{1\}$, $\{2\}$, and $\{3\}$, and all subsets of size 2, namely $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.

The cardinality of $P(S)$ is 2^n when $|S| = n$. This is not such an obvious result, it may be derived in several ways, one of them being the so-called binomial theorem, which says that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j},$$

where $\binom{n}{j}$ counts the number of ways to choose j elements out of n. The notation $\sum_{j=0}^{n}$ means that we sum for the values of j going from 0 to n. See Exercise 33 for a proof of the binomial theorem. When $n = 3$, evaluating in $x = y = 1$, we have

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

and we see that $\binom{3}{0}$ says we pick no element from 3, there is one way, and it corresponds to the empty set, then $\binom{3}{1}$ is telling us that we have 3 ways to choose a single subset, this is for $\{1\}$, $\{2\}$, and $\{3\}$, $\binom{3}{2}$ counts $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\binom{3}{3}$ counts the whole set $\{1,2,3\}$.

When dealing with sets, it is often useful to draw Venn diagrams to show how sets are interacting. They are useful to visualize "unions" and "intersections".

Cardinality

The cardinality |S| *of S* is the number of elements in S.

- e.g. for S={1, 3}, |S| =2

If |S| is finite, S is a finite set; otherwise, S is infinite.

- The set of positive integers is an infinite set.
- The set of prime numbers is an infinite set.
- The set of even prime numbers is a finite set.

- Note: $|\emptyset| = 0$

Power Set

The power set P(S) of a given set S is the set of all subsets of S: P(S) = { A | A $\subseteq$ S}.

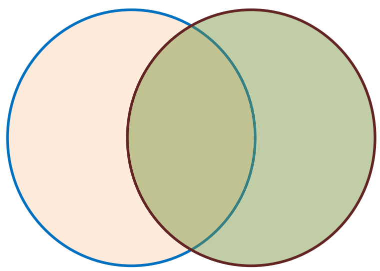
- Example
  - For S= {1,2,3}
    P(S)={$\emptyset$,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}

- If a set A has n elements, then P(s) has 2^n elements.
  - Hint: Try to leverage *the Binomial theorem*

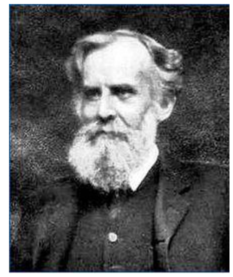
$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

Venn Diagram

A Venn diagram is used to show/visualize the possible relations among a collection of sets.



John Venn
(1834-1923)

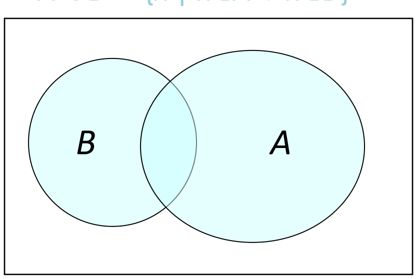


Pictures from wikipedia

Union and Intersection

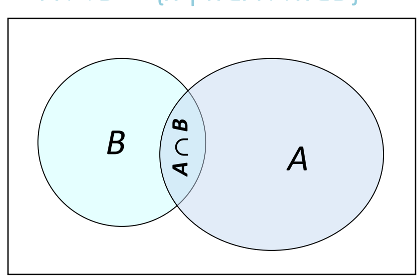
The *union of sets A and B* is the set of those *elements that are either in A or in B, or in both*.

$A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$



The *intersection of the sets A and B* is the set of *all elements that are in both A and B*.

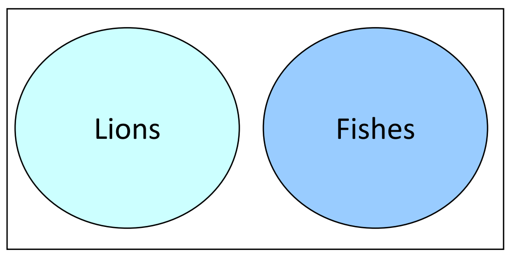
$A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}$



Disjoint Sets

Sets *A* and *B* are *disjoint* iff $A \cap B = \varnothing$

– $|A \cap B| = 0$

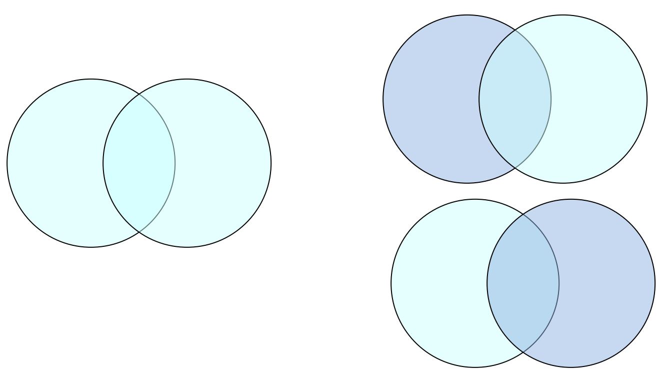


$\text{Lions} \cap \text{Fishes} = \varnothing$

© photographer

Cardinality Of Union

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Definition 30. The union of the sets A and B is by definition

$$A \cup B = \{x,\ x \in A \vee x \in B\}.$$

The intersection of the sets A and B is by definition

$$A \cap B = \{x,\ x \in A \wedge x \in B\}.$$

When the intersection of A and B is empty, we say that A and B are disjoint.

The cardinality of the union and intersection of the sets A and B are related by:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is true, because to count the number of elements in $A \cup B$, we start by counting those in A, and then add those in B. If A and B were disjoint, then we are done, otherwise, we have double counted those in both sets, so we must subtract those in $A \cap B$.

Definition 31. The difference of A and B, also called complement of B with respect to A is the set containing elements that are in B but not in B:

$$A - B = \{x,\ x \in A \wedge x \notin B\}.$$

The complement of A is the complement of A with respect to the universe U:

$$\bar{A} = U - A = \{x,\ x \notin A\}.$$

The universe U is the set that serves as a framework for all our set computations, the biggest set in which all the other sets we are interested in lie. Note that $\overline{\bar{A}} = A$.

Definition 32. The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$:

$$A \times B = \{(a, b),\ a \in A \wedge b \in B\}.$$

Example 46. Take $A = \{1, 2\}$, $B = \{x, y, z\}$. Then

$$A \times B = \{(a, b),\ a \in \{1, 2\} \wedge b \in \{x, y, z\}\}$$

thus a can be either 1 or 2, and for each of these 2 values, b can be either x, y or z:

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}.$$

Note that $A \times B \neq B \times A$, and that a Cartesian product can be formed from n sets $A_1, \ldots, A_n$, which is denoted by $A_1 \times A_2 \times \cdots \times A_n$.

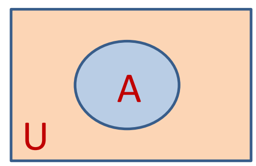
Set Difference & Complement

The difference of A and B (or complement of B with respect to A) is the set containing those elements that are *in A but not in B*.

$$A - B \triangleq \{x \mid x \in A \wedge x \notin B\}$$

The *complement of A* is the complement of A with respect to U.

$$\overline{A} = U - A \triangleq \{x \mid x \notin A\}$$



Cartesian Product

The Cartesian product AxB of the sets A and B is the set of all *ordered pairs (a,b)* where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$$

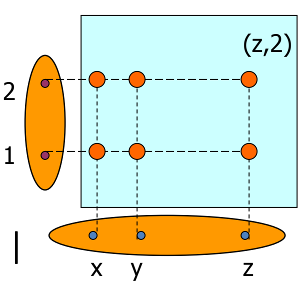
René Descartes (1596-1650)

- Example: A = {1,2}, B = {x,y,z}

  $A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$

  $B \times A = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$

- In general: $A_1 \times A_2 \times \ldots \times A_n \triangleq \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n\}$
- $|A_1 \times A_2 \times \ldots \times A_n| = |A_1|\,|A_2| \ldots |A_n|$



Picture from wikipedia

Definition 33. A collection of nonempty sets $\{A_1, \ldots, A_n\}$ is a partition of a set A if and only if

1. $A = A_1 \cup A_2 \cup \ldots A_n$
2. and $A_1, \ldots, A_n$ are mutually disjoint: $A_i \cap A_j = \varnothing$, $i \neq j$, $i, j = 1, 2, \ldots, n$.

Example 47. Consider $A = \mathbb{Z}$, $A_1 = \{$ even numbers $\}$, $A_2 = \{$ odd numbers $\}$. Then A_1, A_2 form a partition of A.

We next derive a series of set identities:

$$\boxed{A \cap \bar{B} = A - B.}$$

By Definition 31, $A - B = \{x,\ x \in A \wedge x \notin B\}$. Then $A \cap \bar{B} = \{x,\ x \in A \wedge x \in \bar{B}\}$, but by the definition of $\bar{B}$, $A \cap \bar{B} = \{x,\ x \in A \wedge x \notin B\}$, which completes the proof.

We have the set theoretic version of De Morgan's law:

$$\boxed{\overline{A \cap B} = \bar{A} \cup \bar{B}.}$$

We have $\overline{A \cap B} = \{x,\ x \notin A \cap B\} = \{x,\ \neg(x \in A \wedge x \in B)\}$, and using the usual De Morgan's law, we get $\overline{A \cap B} = \{x,\ x \notin A \vee x \notin B\}$ as desired.

Applying de Morgan's law on $\overline{A \cap \bar{B}}$, and $\bar{\bar{B}} = B$ we get:

$$\boxed{\overline{A \cap \bar{B}} = \bar{A} \cup B.}$$

Recall that U denotes the universe set, the one to which belongs all the sets that we are manipulating. In particular, $A \subset U$. We have

$$\boxed{A \cup \varnothing = A,\ A \cap U = A,\ A \cup U = U,\ A \cap \varnothing = \varnothing,\ A \cup A = A,\ A \cap A = A.}$$

Furthermore, the order in which $\cup$ or $\cap$ is done does not matter:

$$\boxed{A \cup B = B \cup A,\ A \cap B = B \cap A,\ A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C.}$$

Distributive laws hold as well:

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).}$$

For example, $A \cap (B \cup C) = \{x,\ x \in A \wedge (x \in B \vee x \in C)\}$ and we can apply the distribute law from propositional logic to get the desired result. And finally

$$\boxed{A \cup (A \cap B) = A,\ A \cap (A \cup B) = A.}$$

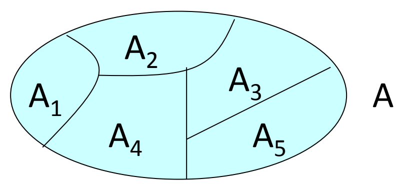
This follows from the fact that $A \cap B$ is a subset of A, while A is a subset of $A \cup B$.

Partition

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A, *iff* $A = A_1 \cup A_2 \cup \cdots \cup A_n$

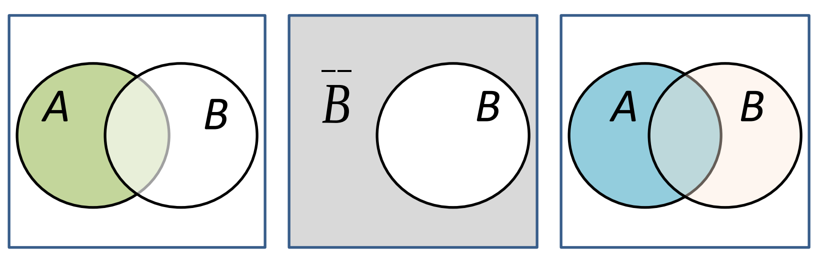
and $A_1, A_2, \ldots, A_n$ are *mutually disjoint*, i.e.

$A_i \cap A_j = \varnothing$ for all $i, j = 1,2,..n$, and $i \neq j$.



Set Identities

$$A \cap \overline{B} = A - B$$

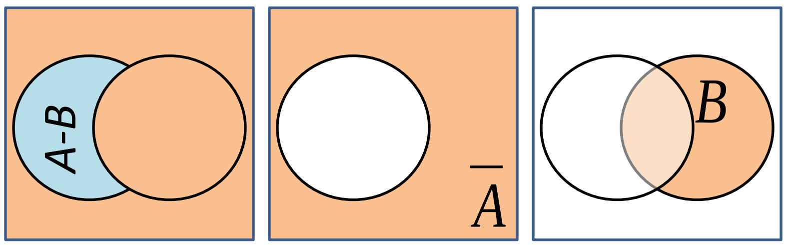


Compare $A \cap \overline{B}$ with $A\text{-}B = \{x \mid x \in A \wedge x \notin B\}$

(not a formal proof)

Set Identities

$$\overline{A\cap\overline{B}}=\overline{A}\cup B$$



- Consider $\overline{A-B} = \overline{A\cap\overline{B}}$
- Apply DeMorgan's Law $\overline{X\cap Y}=\overline{X}\cup\overline{Y}$ with X=A and Y= $\overline{B}$

Set Identities

Identity	Name
$A\cup\emptyset = A$ $A\cap U = A$	Identity laws
$A\cup U = U$ $A\cap\emptyset = \emptyset$	Domination laws
$A\cup A = A$ $A\cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Double Complement laws

Set Identities

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws

Set Identities

Identity	Name
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A - B = A \cap \overline{B}$	Alternate Representation for set difference

Suppose that you want to prove that two sets A and B are equal. We will discuss 3 possible methods to do so:

1. Double inclusion: $A \subseteq B$ and $B \subseteq A$.
2. Set identities.
3. Membership tables.

Example 48. To show that $(B-A) \cup (C-A) = (B \cup C) - A$, we show the double inclusion.

- Take an element $x \in (B-A) \cup (C-A)$, then either $x \in (B-A)$, or $x \in (C-A)$. Then $x \in B \wedge x \notin A$, or $x \in C \wedge x \notin B$. Then either way, $x \in B \cup C \wedge x \notin A$, that is $x \in (B \cup C) - A$, and $(B-A) \cup (C-A) \subseteq (B \cup C) - A$ is shown.
- Now take an element $x \in (B \cup C) - A$, that is $x \in B \cup C$ but $x \notin A$. Then $x \in B$ and not in A, or $x \in C$ and not in A. Then $x \in B - A$ or $x \in C - A$. Thus either way, $x \in (B-A) \cup (C-A)$, which shows that $(B-A) \cup (C-A) \supseteq (B \cup C) - A$

Example 49. We show that $(A-B)-(B-C) = A-B$ using set identities.

$$\begin{aligned} (A-B)-(B-C) &= (A-B) \cap \overline{(B-C)} \\ &= (A \cap \bar{B}) \cap \overline{(B \cap \bar{C})} \\ &= (A \cap \bar{B}) \cap (\bar{B} \cup C) \\ &= [(A \cap \bar{B}) \cap \bar{B}] \cup [(A \cap \bar{B}) \cap C] \end{aligned}$$

where the third equality is De Morgan's law, and the 4rth one is distributivity. We also notice that the first term can be simplified to get $(A \cap \bar{B})$. We then apply distributivity again:

$$(A \cap \bar{B}) \cup [(A \cap \bar{B}) \cap C] = [A \cup [(A \cap \bar{B}) \cap C]] \cap [\bar{B} \cup [(A \cap \bar{B}) \cap C]].$$

Since $(A \cap \bar{B}) \cap C$ is a subset of A, then the first term is A. Similarly, since $(A \cap \bar{B}) \cap C$ is a subset of $\bar{B}$, the second term is $\bar{B}$. Therefore

$$(A-B)-(B-C) = A \cap \bar{B} = A - B.$$

The third method is a membership table, where columns of the table represent different set expressions, and rows take combinations of memberships in constituent sets: 1 means membership, and 0 non-membership. For two sets to be equal, they need to have identical columns.

Proving Set Equality

- Recall. Two sets are equal *if and only if* they contain exactly the same elements, i.e., ***iff*** **A⊆B *and* B⊆A**

- Three methods to prove set equality:
  - Show that each set is a subset of the other
  - Apply set identities theorems
  - Use membership table

Each Others' Subset

Show that (B-A)∪ (C-A)=(B ∪ C)-A.

For any *x* ∈LHS, *x* ∈(*B*-*A*) or *x* ∈(*C*-*A*) [or both].

when $x \in B - A$
$$\Longrightarrow (x \in B) \wedge (x \notin A)$$
$$\Longrightarrow (x \in B \cup C) \wedge (x \notin A)$$
$$\Longrightarrow x \in (B \cup C) - A$$

when $x \in C - A$
$$\Longrightarrow (x \in C) \wedge (x \notin A)$$
$$\Longrightarrow (x \in B \cup C) \wedge (x \notin A)$$
$$\Longrightarrow x \in (B \cup C) - A$$

Therefore, LHS ⊆ RHS

Each Others' Subset

Show that (B-A)$\cup$ (C-A)=(B $\cup$ C)-A.

For any $x \in$RHS, $x \in (B \cup C)$ and $x \notin$A.

when $x \in B$ and $x \notin A$

$$(x \in B) \wedge (x \notin A) \Longrightarrow x \in B - A$$
$$\Longrightarrow x \in (B-A) \cup (C-A)$$

when $x \in C$ and $x \notin A$,

$$(x \in C) \wedge (x \notin A) \Longrightarrow x \in C - A$$
$$\Longrightarrow x \in (B-A) \cup (C-A)$$

Therefore, RHS $\subseteq$ LHS

With LHS $\subseteq$ RHS and RHS $\subseteq$ LHS, we can conclude that **LHS = RHS**

Using Set Identities

Show that (A-B)-(B-C)=A-B

$$(A-B)-(B-C) = (A \cap \overline{B}) \cap \overline{(B \cap \overline{C})}$$ (By alternate representation for set difference)

$$= (A \cap \overline{B}) \cap (\overline{B} \cup C)$$ (By De Morgan's laws)

$$= [(A \cap \overline{B}) \cap \overline{B}] \cup [(A \cap \overline{B}) \cap C]$$ (By Distributive laws)

$$= [A \cap (\overline{B} \cap \overline{B})] \cup [A \cap (\overline{B} \cap C)]$$ (By Associative laws)

$$= (A \cap \overline{B}) \cup [A \cap (\overline{B} \cap C)]$$ (By Idempotent laws)

$$= A \cap [\overline{B} \cup (\overline{B} \cap C)]$$ (By Distributive laws)

$$= A \cap \overline{B}$$ (By Absorption laws)

$$= A - B$$ (By the alternate representation for set difference)

Using Membership Tables

Similar to truth table (in propositional logic)

- Columns for different set expressions
- Rows for all combinations of memberships in constituent sets
- *"1" = membership* , *"0" =non-membership*
- Two sets are equal, *iff* they have identical columns

Prove that $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Example 50. To prove $(A \cup B) - B = A - B$, we create a table

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0			
0	1			
1	0			
1	1			

The first row, if x is not in A and not in B, it will not be in any of the sets, therefore the first row contains only zeroes. If x is only in B, then it belongs to $A \cup B$, but not in the others, since B is removed. So the second row has only a 1 in $A \cup B$. Then if x is only in A, it belongs to all the three sets. Finally, if x is in both A and B, it is in their intersection, therefore it belongs to $A \cup B$, but not in the 2 others, since B is removed.

Exercises for Chapter 4

Exercise 33. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},\ n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Exercise 34. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B),\ P(A),\ P(A \cup B),\ P(A \times B).$$

Exercise 35. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

Exercise 36. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

Exercise 37. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 38. Show that the empty set is a subset of all non-null sets.

Exercise 39. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Exercise 40. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 41. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 42. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 43. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 44. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Exercise 45. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?